M. Rahman

Integral Equations and their Applications

$$u_n(x) = \sum_{m=0}^{n} \lambda^m \psi_m(x)$$

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x e^{x-t} \sum_{n=0}^{\infty} u_n(t)dt$$

$$A = -\int_a^x \frac{y_2(\eta)f(\eta)}{W(\eta)} d\eta$$

$$u_n = b + \int_a^x f(x, u_n$$

$$u_2(x) = f(x) + \lambda \int_a^b K(x,t)u_1(t)dt$$

$$y = -y_1(x)\int_a^x \frac{y_2(\eta)f(\eta)}{W(\eta)} d\eta - y_2(x)\int_x^b \frac{y_1(\eta)f(\eta)}{W(\eta)} d\eta$$

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Contents

Preface

While scientists and engineers can already choose from a number of books on integral equations, this new book encompasses recent developments including some preliminary backgrounds of formulations of integral equations governing the physical situation of the problems. It also contains elegant analytical and numerical methods, and an important topic of the variational principles. This book is primarily intended for the senior undergraduate students and beginning graduate students of engineering and science courses. The students in mathematical and physical sciences will find many sections of divert relevance. The book contains eight chapters. The chapters in the book are pedagogically organized. This book is specially designed for those who wish to understand integral equations without having extensive mathematical background. Some knowledge of integral calculus, ordinary differential equations, partial differential equations, Laplace transforms, Fourier transforms, Hilbert transforms, analytic functions of complex variables and contour integrations are expected on the part of the reader.

The book deals with linear integral equations, that is, equations involving an unknown function which appears under an integral sign. Such equations occur widely in diverse areas of applied mathematics and physics. They offer a powerful technique for solving a variety of practical problems. One obvious reason for using the integral equation rather than differential equations is that all of the conditions specifying the initial value problems or boundary value problems for a differential equation can often be condensed into a single integral equation. In the case of partial differential equations, the dimension of the problem is reduced in this process so that, for example, a boundary value problem for a partial differential equation in two independent variables transform into an integral equation involving an unknown function of only one variable. This reduction of what may represent a complicated mathematical model of a physical situation into a single equation is itself a significant step, but there are other advantages to be gained by replacing differentiation with integration. Some of these advantages arise because integration is a smooth process, a feature which has significant implications when approximate solutions are sought. Whether one is looking for an exact solution to a given problem or having to settle for an approximation to it, an integral equation formulation can often provide a useful way forward. For this reason integral equations have attracted attention for

most of the last century and their theory is well-developed.

While I was a graduate student at the Imperial College's mathematics department during 1966-1969, I was fascinated with the integral equations course given by Professor Rosenblatt. His deep knowledge about the subject impressed me and gave me a love for integral equations. One of the aims of the course given by Professor Rosenblatt was to bring together students from pure mathematics and applied mathematics, often regarded by the students as totally unconnected. This book contains some theoretical development for the pure mathematician but these theories are illustrated by practical examples so that an applied mathematician can easily understand and appreciate the book.

This book is meant for the senior undergraduate and the first year postgraduate student. I assume that the reader is familiar with classical real analysis, basic linear algebra and the rudiments of ordinary differential equation theory. In addition, some acquaintance with functional analysis and Hilbert spaces is necessary, roughly at the level of a first year course in the subject, although I have found that a limited familiarity with these topics is easily considered as a bi-product of using them in the setting of integral equations. Because of the scope of the text and emphasis on practical issues, I hope that the book will prove useful to those working in application areas who find that they need to know about integral equations.

I felt for many years that integral equations should be treated in the fashion of this book and I derived much benefit from reading many integral equation books available in the literature. Others influence in some cases by acting more in spirit, making me aware of the sort of results we might seek, papers by many prominent authors. Most of the material in the book has been known for many years, although not necessarily in the form in which I have presented it, but the later chapters do contain some results I believe to be new.

Digital computers have greatly changed the philosophy of mathematics as applied to engineering. Many applied problems that cannot be solved explicitly by analytical methods can be easily solved by digital computers. However, in this book I have attempted the classical analytical procedure. There is too often a gap between the approaches of a pure and an applied mathematician to the same problem, to the extent that they may have little in common. I consider this book a middle road where I develop, the general structures associated with problems which arise in applications and also pay attention to the recovery of information of practical interest. I did not avoid substantial matters of calculations where these are necessary to adapt the general methods to cope with classes of integral equations which arise in the applications. I try to avoid the rigorous analysis from the pure mathematical view point, and I hope that the pure mathematician will also be satisfied with the dealing of the applied problems.

The book contains eight chapters, each being divided into several sections. In this text, we were mainly concerned with linear integral equations, mostly of second-kind. Chapter 1 introduces the classifications of integral equations and necessary techniques to convert differential equations to integral equations or vice versa. Chapter 2 deals with the linear Volterra integral equations and the relevant solution techniques. Chapter 3 is concerned with the linear Fredholme integral equations

and also solution techniques. Nonlinear integral equations are investigated in Chapter 4. Adomian decomposition method is used heavily to determine the solution in addition to other classical solution methods. Chapter 5 deals with singular integral equations along with the variational principles. The transform calculus plays an important role in this chapter. Chapter 6 introduces the integro-differential equations. The Volterra and Fredholm type integro-differential equations are successfully manifested in this chapter. Chapter 7 contains the orthogonal systems of functions. Green's functions as the kernel of the integral equations are introduced using simple practical problems. Some practical problems are solved in this chapter. Chapter 8 deals with the applied problems of advanced nature such as arising in ocean waves, seismic response, transverse oscillations and flows of heat. The book concludes with four appendices.

In this computer age, classical mathematics may sometimes appear irrelevant. However, use of computer solutions without real understanding of the underlying mathematics may easily lead to gross errors. A solid understanding of the relevant mathematics is absolutely necessary. The central topic of this book is integral equations and the calculus of variations to physical problems. The solution techniques of integral equations by analytical procedures are highlighted with many practical examples.

For many years the subject of functional equations has held a prominent place in the attention of mathematicians. In more recent years this attention has been directed to a particular kind of functional equation, an integral equation, wherein the unknown function occurs under the integral sign. The study of this kind of equation is sometimes referred to as the inversion of a definite integral.

In the present book I have tried to present in readable and systematic manner the general theory of linear integral equations with some of its applications. The applications given are to differential equations, calculus of variations, and some problems which lead to differential equations with boundary conditions. The applications of mathematical physics herein given are to Neumann's problem and certain vibration problems which lead to differential equations with boundary conditions. An attempt has been made to present the subject matter in such a way as to make the book suitable as a text on this subject in universities.

The aim of the book is to present a clear and well-organized treatment of the concept behind the development of mathematics and solution techniques. The text material of this book is presented in a highly readable, mathematically solid format. Many practical problems are illustrated displaying a wide variety of solution techniques.

There are more than 100 solved problems in this book and special attention is paid to the derivation of most of the results in detail, in order to reduce possible frustrations to those who are still acquiring the requisite skills. The book contains approximately 150 exercises. Many of these involve extension of the topics presented in the text. Hints are given in many of these exercises and answers to some selected exercises are provided in Appendix C. The prerequisites to understand the material contained in this book are advanced calculus, vector analysis and techniques of solving elementary differential equations. Any senior undergraduate student who

has spent three years in university, will be able to follow the material contained in this book. At the end of most of the chapters there are many exercises of practical interest demanding varying levels of effort.

While it has been a joy to write this book over a number of years, the fruits of this labor will hopefully be in learning of the enjoyment and benefits realized by the reader. Thus the author welcomes any suggestions for the improvement of the text.

M. Rahman
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1 Introduction

1.1 Preliminary concept of the integral equation

An integral equation is defined as an equation in which the unknown function $u(x)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations (Refs. [2], [3] and [6]).

The development of science has led to the formation of many physical laws, which, when restated in mathematical form, often appear as differential equations. Engineering problems can be mathematically described by differential equations, and thus differential equations play very important roles in the solution of practical problems. For example, Newton's law, stating that the rate of change of the momentum of a particle is equal to the force acting on it, can be translated into mathematical language as a differential equation. Similarly, problems arising in electric circuits, chemical kinetics, and transfer of heat in a medium can all be represented mathematically as differential equations.

A typical form of an integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt \tag{1.1}$$

where $K(x,t)$ is called the kernel of the integral equation (1.1), and $\alpha(x)$ and $\beta(x)$ are the limits of integration. It can be easily observed that the unknown function $u(x)$ appears under the integral sign. It is to be noted here that both the kernel $K(x,t)$ and the function $f(x)$ in equation (1.1) are given functions; and λ is a constant parameter. The prime objective of this text is to determine the unknown function $u(x)$ that will satisfy equation (1.1) using a number of solution techniques. We shall devote considerable efforts in exploring these methods to find solutions of the unknown function.

1.2 Historical background of the integral equation

In 1825 Abel, an Italian mathematician, first produced an integral equation in connection with the famous *tautochrone* problem (see Refs. [1], [4] and [5]). The problem is connected with the determination of a curve along which a heavy particle, sliding without friction, descends to its lowest position, or more generally, such that the time of descent is a given function of its initial position. To be more specific, let us consider a smooth curve situated in a vertical plane. A heavy particle starts from rest at any position P (see Figure 1.1).

Let us find, under the action of gravity, the time T of descent to the lowest position O. Choosing O as the origin of the coordinates, the x-axis vertically upward, and the y-axis horizontal. Let the coordinates of P be (x, y), of Q be (ξ, η), and s the arc OQ.

At any instant, the particle will attain the potential energy and kinetic energy at Q such that the sum of which is constant, and mathematically it can be stated as

$$
\begin{aligned}
K.E. + P.E. &= \text{constant} \\
\tfrac{1}{2}mv^2 + mg\xi &= \text{constant} \\
\text{or} \quad \tfrac{1}{2}v^2 + g\xi &= \text{C}
\end{aligned}
\tag{1.2}
$$

where m is the mass of the particle, $v(t)$ the speed of the particle at Q, g the acceleration due to gravity, and ξ the vertical coordinate of the particle at Q. Initially, $v(0) = 0$ at P, the vertical coordinate is x, and hence the constant C can be determined as $\text{C} = gx$.

Thus, we have

$$
\begin{aligned}
\tfrac{1}{2}v^2 + g\xi &= gx \\
v^2 &= 2g(x - \xi) \\
v &= \pm\sqrt{2g(x - \xi)}
\end{aligned}
\tag{1.3}
$$

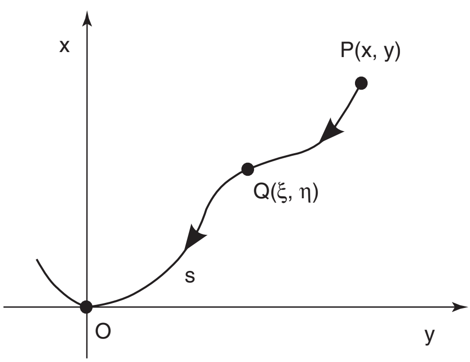

Figure 1.1: Schematic diagram of a smooth curve.

But $v = \frac{ds}{dt}$ = speed along the curve s. Therefore,

$$\frac{ds}{dt} = \pm\sqrt{2g(x-\xi)}.$$

Considering the negative value of $\frac{ds}{dt}$ and integrating from P to Q by separating the variables, we obtain

$$\begin{aligned}\int_P^Q dt &= -\int_P^Q \frac{ds}{\sqrt{2g(x-\xi)}}\\ t &= -\int_P^Q \frac{ds}{\sqrt{2g(x-\xi)}}\end{aligned}$$

The total time of descent is, then,

$$\begin{aligned}\int_P^O dt &= -\int_P^O \frac{ds}{\sqrt{2g(x-\xi)}}\\ T &= \int_O^P \frac{ds}{\sqrt{2g(x-\xi)}} \qquad (1.4)\end{aligned}$$

If the shape of the curve is given, then s can be expressed in terms of ξ and hence ds can be expressed in terms of ξ. Let $ds = u(\xi)d\xi$, the equation (1.4) takes the form

$$T = \int_0^x \frac{u(\xi)d\xi}{\sqrt{2g(x-\xi)}}$$

Abel set himself the problem of finding the curve for which the time T of descent is a given function of x, say $f(x)$. Our problem, then, is to find the unknown function $u(x)$ from the equation

$$\begin{aligned}f(x) &= \int_0^x \frac{u(\xi)d\xi}{\sqrt{2g(x-\xi)}}\\ &= \int_0^x K(x,\xi)u(\xi)d\xi. \qquad (1.5)\end{aligned}$$

This is a linear integral equation of the first kind for the determination of $u(x)$. Here, $K(x,\xi) = \frac{1}{\sqrt{2g(x-\xi)}}$ is the kernel of the integral equation. Abel solved this problem already in 1825, and in essentially the same manner which we shall use; however, he did not realize the general importance of such types of functional equations.

1.3 An illustration from mechanics

The differential equation which governs the mass-spring system is given by (see Figure 1.2)

$$m\frac{d^2u}{dt^2} + ku = f(t) \quad (0 \le t < \infty)$$

with the initial conditions, $u(0) = u_0$, and $\frac{du}{dt} = \dot{u}_0$, where k is the stiffness of the string, $f(t)$ the prescribed applied force, u_0 the initial displacement, and $\dot{u}_0$ the initial value. This problem can be easily solved by using the Laplace transform. We transform this ODE problem into an equivalent integral equation as follows:

Integrating the ODE with respect to t from 0 to t yields

$$m\frac{du}{dt} - m\dot{u}_0 + k\int_0^t u(\tau)d\tau = \int_0^t f(\tau)d\tau.$$

Integrating again gives

$$mu(t) - mu_0 - m\dot{u}_0 t + k\int_0^t \int_0^t u(\tau)d\tau d\tau = \int_0^t \int_0^t f(\tau)d\tau d\tau. \tag{1.6}$$

We know that if $y(t) = \int_0^t \int_0^t u(\tau)d\tau d\tau$, then $\mathcal{L}\{y(t)\} = \mathcal{L}\{\int_0^t \int_0^t f(\tau)d\tau d\tau\} = \frac{1}{s^2}\mathcal{L}\{u(t)\}$. Therefore, by using the convolution theorem, the Laplace inverse is obtained as $y(t) = \int_0^t (t-\tau)u(\tau)d\tau$, which is known as the convolution integral. Hence using the convolution property, equation (1.6) can be written as

$$u(t) = u_0 + \dot{u}_0 t + \frac{1}{m}\int_0^t (t-\tau)f(\tau)d\tau - \frac{k}{m}\int_0^t (t-\tau)u(\tau)d\tau, \tag{1.7}$$

which is an integral equation. Unfortunately, this is not the solution of the original problem, because the presence of the unknown function $u(t)$ under the integral sign. Rather, it is an example of an integral equation because of the presence of the unknown function within the integral. Beginning with the integral equation, it is possible to reverse our steps with the help of the Leibnitz rule, and recover the original system, so that they are equivalent. In the present illustration, the physics (namely, Newton's second law) gave us the differential equation of motion, and it

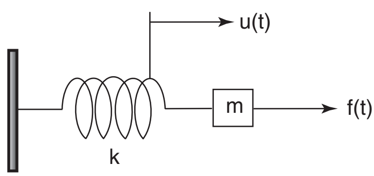

Figure 1.2: Mass spring system.

was only by manipulation, we obtained the integral equation. In the Abel's problem, the physics gave us the integral equation directly. In any event, observe that we can solve the integral equation by application of the Laplace Transform. Integral equations of the convolution type can easily be solved by the Laplace transform .

1.4 Classification of integral equations

An integral equation can be classified as a linear or nonlinear integral equation as we have seen in the ordinary and partial differential equations. In the previous section, we have noticed that the differential equation can be equivalently represented by the integral equation. Therefore, there is a good relationship between these two equations.

The most frequently used integral equations fall under two major classes, namely **Volterra** and **Fredholm** integral equations. Of course, we have to classify them as homogeneous or nonhomogeneous; and also linear or nonlinear. In some practical problems, we come across singular equations also.

In this text, we shall distinguish four major types of integral equations – the two main classes and two related types of integral equations. In particular, the four types are given below:

- Volterra integral equations
- Fredholm integral equations
- Integro-differential equations
- Singular integral equations

We shall outline these equations using basic definitions and properties of each type.

1.4.1 Volterra integral equations

The most standard form of Volterra linear integral equations is of the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt \tag{1.8}$$

where the limits of integration are function of x and the unknown function $u(x)$ appears linearly under the integral sign. If the function $\phi(x) = 1$, then equation (1.8) simply becomes

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt \tag{1.9}$$

and this equation is known as the Volterra integral equation of the second kind; whereas if $\phi(x) = 0$, then equation (1.8) becomes

$$f(x) + \lambda \int_a^x K(x,t)u(t)dt = 0 \tag{1.10}$$

which is known as the Volterra equation of the first kind.

1.4.2 Fredholm integral equations

The most standard form of Fredholm linear integral equations is given by the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{1.11}$$

where the limits of integration a and b are constants and the unknown function $u(x)$ appears linearly under the integral sign. If the function $\phi(x) = 1$, then (1.11) becomes simply

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{1.12}$$

and this equation is called Fredholm integral equation of second kind; whereas if $\phi(x) = 0$, then (1.11) yields

$$f(x) + \lambda \int_a^b K(x,t)u(t)dt = 0 \tag{1.13}$$

which is called Fredholm integral equation of the first kind.

Remark

It is important to note that integral equations arise in engineering, physics, chemistry, and biological problems. Many initial and boundary value problems associated with the ordinary and partial differential equations can be cast into the integral equations of Volterra and Fredholm types, respectively.

If the unknown function $u(x)$ appearing under the integral sign is given in the functional form $F(u(x))$ such as the power of $u(x)$ is no longer unity, e.g. $F(u(x)) = u^n(x), n \neq 1$, or $\sin u(x)$ etc., then the Volterra and Fredholm integral equations are classified as nonlinear integral equations. As for examples, the following integral equations are nonlinear integral equations:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u^2\,(t)\, dt$$

$$u(x) = f(x) + \lambda \int_a^x K(x,t) \sin\,(u(t))\, dt$$

$$u(x) = f(x) + \lambda \int_a^x K(x,t) \ln\,(u(t))\, dt$$

Next, if we set $f(x) = 0$, in Volterra or Fredholm integral equations, then the resulting equation is called a homogeneous integral equation, otherwise it is called nonhomogeneous integral equation.

1.4.3 Singular integral equations

A singular integral equation is defined as an integral with the infinite limits or when the kernel of the integral becomes unbounded at a certain point in the interval. As for examples,

$$
\begin{aligned}
u(x) &= f(x) + \lambda \int_{-\infty}^{\infty} u(t)dt \\
f(x) &= \int_0^x \frac{1}{(x-t)^{\alpha}} u(t)dt, \quad 0 < \alpha < 1
\end{aligned} \tag{1.14}
$$

are classified as the singular integral equations.

1.4.4 Integro-differential equations

In the early 1900, Vito Volterra studied the phenomenon of population growth, and new types of equations have been developed and termed as the integro-differential equations. In this type of equations, the unknown function $u(x)$ appears as the combination of the ordinary derivative and under the integral sign. In the electrical engineering problem, the current $I(t)$ flowing in a closed circuit can be obtained in the form of the following integro-differential equation,

$$
L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\tau)d\tau = f(t), \quad I(0) = I_0 \tag{1.15}
$$

where L is the inductance, R the resistance, C the capacitance, and $f(t)$ the applied voltage. Similar examples can be cited as follows:

$$
u''(x) = f(x) + \lambda \int_0^x (x-t)u(t)dt, \quad u(0) = 0, u'(0) = 1, \tag{1.16}
$$

$$
u'(x) = f(x) + \lambda \int_0^1 (xt)u(t)dt, \quad u(0) = 1. \tag{1.17}
$$

Equations (1.15) and (1.16) are of Volterra type integro-differential equations, whereas equation (1.17) Fredholm type integro-differential equations. These terminologies were concluded because of the presence of indefinite and definite integrals.

1.5 Converting Volterra equation to ODE

In this section, we shall present the technique that converts Volterra integral equations of second kind to equivalent ordinary differential equations. This may be

achieved by using the Leibnitz rule of differentiating the integral $\int_{a(x)}^{b(x)} F(x,t)dt$ with respect to x, we obtain

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(x,t)dt = \int_{a(x)}^{b(x)} \frac{\partial F(x,t)}{\partial x}dt + \frac{db(x)}{dx}F(x,b(x)) - \frac{da(x)}{dx}F(x,a(x)), \tag{1.18}$$

where $F(x,t)$ and $\frac{\partial F}{\partial x}(x,t)$ are continuous functions of x and t in the domain $\alpha \leq x \leq \beta$ and $t_0 \leq t \leq t_1$; and the limits of integration $a(x)$ and $b(x)$ are defined functions having continuous derivatives for $\alpha \leq x \leq \beta$. For more information the reader should consult the standard calculus book including Rahman (2000). A simple illustration is presented below:

$$\begin{aligned}\frac{d}{dx}\int_0^x \sin(x-t)u(t)dt &= \int_0^x \cos(x-t)u(t)dt + \left(\frac{dx}{dx}\right)(\sin(x-x)u(x)) \\ &\quad - \left(\frac{d0}{dx}\right)(\sin(x-0)u(0)) \\ &= \int_0^x \cos(x-t)u(t)dt.\end{aligned}$$

1.6 Converting IVP to Volterra equations

We demonstrate in this section how an initial value problem (IVP) can be transformed to an equivalent Volterra integral equation. Let us consider the integral equation

$$y(t) = \int_0^t f(t)dt \tag{1.19}$$

The Laplace transform of $f(t)$ is defined as $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = F(s)$. Using this definition, equation (1.19) can be transformed to

$$\mathcal{L}\{y(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\}.$$

In a similar manner, if $y(t) = \int_0^t \int_0^t f(t)dtdt$, then

$$\mathcal{L}\{y(t)\} = \frac{1}{s^2}\mathcal{L}\{f(t)\}.$$

This can be inverted by using the convolution theorem to yield

$$y(t) = \int_0^t (t-\tau)f(\tau)d\tau.$$

If

$$y(t) = \underbrace{\int_0^t \int_0^t \cdots \int_0^t f(t)dtdt\cdots dt}_{n\text{-fold integrals}}$$

then $\mathcal{L}\{y(t)\} = \frac{1}{s^n}\mathcal{L}\{f(t)\}$. Using the convolution theorem, we get the Laplace inverse as

$$y(t) = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)d\tau.$$

Thus the n-fold integrals can be expressed as a single integral in the following manner:

$$\underbrace{\int_0^t \int_0^t \cdots \int_0^t f(t)dtdt\cdots dt}_{n\text{-fold integrals}} = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)d\tau. \tag{1.20}$$

This is an essential and useful formula that has enormous applications in the integral equation problems.

1.7 Converting BVP to Fredholm integral equations

In the last section we have demonstrated how an IVP can be transformed to an equivalent Volterra integral equation. We present in this section how a boundary value problem (BVP) can be converted to an equivalent Fredholm integral equation. The method is similar to that discussed in the previous section with some exceptions that are related to the boundary conditions. It is to be noted here that the method of reducing a BVP to a Fredholm integral equation is complicated and rarely used. We demonstrate this method with an illustration.

Example 1.1

Let us consider the following second-order ordinary differential with the given boundary conditions.

$$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x) \tag{1.21}$$

with the boundary conditions

$$\begin{aligned} x = a: &\quad y(a) = \alpha \\ y = b: &\quad y(b) = \beta \end{aligned} \tag{1.22}$$

where α and β are given constants. Let us make transformation

$$y''(x) = u(x) \tag{1.23}$$

Integrating both sides of equation (1.23) from a to x yields

$$y'(x) = y'(a) + \int_a^x u(t)dt \tag{1.24}$$

Note that $y'(a)$ is not prescribed yet. Integrating both sides of equation (1.24) with respect to x from a to x and applying the given boundary condition at $x = a$, we find

$$\begin{aligned} y(x) &= y(a) + (x-a)y'(a) + \int_a^x \int_a^x u(t)dtdt \\ &= \alpha + (x-a)y'(a) + \int_a^x \int_a^x u(t)dtdt \end{aligned} \tag{1.25}$$

and using the boundary condition at $x = b$ yields

$$y(b) = \beta = \alpha + (b-a)y'(a) + \int_a^b \int_a^b u(t)dtdt,$$

and the unknown constant $y'(a)$ is determined as

$$y'(a) = \frac{\beta - \alpha}{b-a} - \frac{1}{b-a}\int_a^b \int_a^b u(t)dtdt. \tag{1.26}$$

Hence the solution (1.25) can be rewritten as

$$\begin{aligned} y(x) = \alpha + (x-a)&\left\{\frac{\beta-\alpha}{b-a} - \frac{1}{b-a}\int_a^b \int_a^b u(t)dtdt\right\} \\ &+ \int_a^x \int_a^x u(t)dtdt \end{aligned} \tag{1.27}$$

Therefore, equation (1.21) can be written in terms of $u(x)$ as

$$\begin{aligned} u(x) = f(x) - P(x)&\left\{y'(a) + \int_a^x u(t)dt\right\} \\ &-Q(x)\left\{\alpha + (x-a)y'(a) + \int_a^x \int_a^x u(t)dtdt\right\} \end{aligned} \tag{1.28}$$

where $u(x) = y''(x)$ and so $y(x)$ can be determined, in principle, from equation (1.27). This is a complicated procedure to determine the solution of a BVP by equivalent Fredholm integral equation.

A special case

If $a = 0$ and $b = 1$, i.e. $0 \le x \le 1$, then

$$
\begin{aligned}
y(x) &= \alpha + xy'(0) + \int_0^x \int_0^x u(t)dtdt \\
&= \alpha + xy'(a) + \int_0^x (x-t)u(t)dt
\end{aligned}
$$

And hence the unknown constant $y'(0)$ can be determined as

$$
\begin{aligned}
y'(0) &= (\beta - \alpha) - \int_0^1 (1-t)u(t)dt \\
&= (\beta - \alpha) - \int_0^x (1-t)u(t)dt - \int_x^1 (1-t)u(t)dt
\end{aligned}
$$

And thus we have

$$
\begin{aligned}
u(x) &= f(x) - P(x)\left\{y'(0) + \int_0^x u(t)dt\right\} \\
&\quad -Q(x)\left\{\alpha + xy'(0) + \int_0^x (x-t)u(t)dt\right\} \\
u(x) &= f(x) - (\beta-\alpha)(P(x) + xQ(x)) - \alpha Q(x) + \int_0^1 K(x,t)u(t)dt
\end{aligned}
\tag{1.29}
$$

where the kernel $K(x,t)$ is given by

$$
K(x,t) = \begin{cases} (P(x) + tQ(x))(1-x) & 0 \le t \le x \\ (P(x) + xQ(x))(1-t) & x \le t \le 1 \end{cases} \tag{1.30}
$$

It can be easily verified that $K(x,t) = K(t,x)$ confirming that the kernel is symmetric. The Fredholm integral equation is given by (1.29).

Example 1.2

Let us consider the following boundary value problem.

$$
\begin{aligned}
y''(x) &= f(x, y(x)), \quad 0 \le x \le 1 \\
y(0) &= y_0, \quad y(1) = y_1
\end{aligned}
\tag{1.31}
$$

Integrating equation (1.31) with respect to x from 0 to x two times yields

$$
\begin{aligned}
y(x) &= y(0) + xy'(0) + \int_0^x \int_0^x f(t, y(t))dtdt \\
&= y_0 + xy'(0) + \int_0^x (x-t)f(t, y(t))dt
\end{aligned}
\tag{1.32}
$$

To determine the unknown constant $y'(0)$, we use the condition at $x=1$, i.e. $y(1)=y_1$. Hence equation (1.32) becomes

$$y(1) = y_1 = y_0 + y'(0) + \int_0^1 (1-t)f(t, y(t))dt,$$

and the value of $y'(0)$ is obtained as

$$y'(0) = (y_1 - y_0) - \int_0^1 (1-t)f(t, y(t))dt.$$

Thus, equation (1.32) can be written as

$$y(x) = y_0 + x(y_1 - y_0) - \int_0^1 K(x,t)f(t, y(t))dt, \quad 0 \le x \le 1 \tag{1.33}$$

in which the kernel is given by

$$K(x,t) = \begin{cases} t(1-t) & 0 \le t \le x \\ x(1-t) & x \le t \le 1. \end{cases} \tag{1.34}$$

Once again we can reverse the process and deduce that the function y which satisfies the integral equation also satisfies the BVP. If we now specialize equation (1.31) to the simple linear BVP $y''(x) = -\lambda y(x)$, $0 < x < 1$ with the boundary conditions $y(0) = y_0, y(1) = y_1$, then equation (1.33) reduces to the second kind Fredholm integral equation

$$y(x) = F(x) + \lambda \int_0^1 K(x,t)y(t)dt, \quad 0 \le x \le 1$$

where $F(x) = y_0 + x(y_1 - y_0)$. It can be easily verified that $K(x,t) = K(t,x)$ confirming that the kernel is symmetric.

Example 1.3

As the third example, consider the following boundary value problem

$$\begin{aligned} y''(x) + y(x) &= x, \quad 0 < x < \pi/2 \\ y(0) &= 1, \quad y(\pi/2) = \pi \end{aligned} \tag{1.35}$$

The analytical solution of the above problem is simply $y(x) = \cos x + \frac{\pi}{2}\sin x + x$. We want to reduce it into Fredholm integral equation.

Integrating the differential equation with respect to x from 0 to x twice and using the boundary conditions, we obtain

$$\begin{aligned} y(x) &= xy'(0) + \frac{x^3}{6} - \int_0^x \int_0^x y(t)dtdt \\ &= xy'(0) + \frac{x^3}{6} - \int_0^x (x-t)y(t)dt \end{aligned}$$

Using the boundary condition at $x = \frac{\pi}{2}$, the unknown constant $y'(0)$ can be obtained as

$$y'(0) = 2 - \frac{\pi^2}{24} + \frac{2}{\pi}\int_0^{\pi/2} (\pi/2 - t)y(t)dt.$$

With this information the equation for $y(x)$ can be put in the form of Fredholm integral equation

$$y(x) = f(x) + \int_0^{\pi/2} K(x,t)y(t)dt,$$

where $f(x) = 2x - \frac{\pi^2}{24}x + \frac{x^3}{6}$ and the kernel is given by

$$K(x,t) = \begin{cases} \frac{2t}{\pi}(\pi/2 - t) & 0 \le t \le x \\ \frac{2x}{\pi}(\pi/2 - x) & x \le t \le \pi/2, \end{cases} \tag{1.36}$$

which can be easily shown that the kernel is symmetric as before.

1.8 Types of solution techniques

There are a host of solution techniques that are available to solve the integral equations. Two important traditional methods are the method of successive approximations and the method of successive substitutions. In addition, the series method and the direct computational method are also suitable for some problems. The recently developed methods, namely the Adomian decomposition method (ADM) and the modified decomposition method, are gaining popularity among scientists and engineers for solving highly nonlinear integral equations. Singular integral equations encountered by Abel can easily be solved by using the Laplace transform method. Volterra integral equations of convolution type can be solved using the Laplace transform method. Finally, for nonlinear problems, numerical techniques will be of extremely useful to solve the highly complicated problems.

This textbook will contain two chapters dealing with the integral equations applied to classical problems and the modern advanced problems of physical interest.

1.9 Exercises

1. Classify each of the following integral equations as Volterra or Fredholm integral equation, linear or nonlinear, and homogeneous or nonhomogeneous:

$$
\begin{aligned}
(a)\quad & u(x) = x + \int_0^1 (x-t)^2 u(t)dt \\
(b)\quad & u(x) = e^x + \int_0^x t^2 u^2(t)dt \\
(c)\quad & u(x) = \cos x + \int_0^{\pi/2} \cos xu(t)dt \\
(d)\quad & u(x) = 1 + \frac{x}{4}\int_0^1 \frac{1}{x+t}\frac{1}{u(t)}dt
\end{aligned}
$$

2. Classify each of the following integro-differential equations as Volterra integro-differential equations or Fredholm integro-differential equations. Also determine whether the equation is linear or nonlinear.

$$
\begin{aligned}
(a)\quad & u'(x) = 1 + \int_0^x e^{-2t} u^3(t)dt, \quad u(0) = 1 \\
(b)\quad & u''(x) = \frac{x^2}{2} - \int_0^x (x-t)u^2(t)dt, \quad u(0) = 1, u'(0) = 0 \\
(c)\quad & u'''(x) = \sin x - x + \int_0^{\pi/2} xtu'(t)dt \\
& u(0) = 1, u'(0) = 0, u''(0) = -1
\end{aligned}
$$

3. Integrate both sides of each of the following differential equations once from 0 to x, and use the given initial conditions to convert to a corresponding integral equations or integro-differential equations.

$$
\begin{aligned}
(a)\quad & u'(x) = u^2(x), \quad u(0) = 4 \\
(b)\quad & u''(x) = 4xu^2(x), \quad u(0) = 2, u'(0) = 1 \\
(c)\quad & u''(x) = 2xu(x), \quad u(0) = 0, u'(0) = 1.
\end{aligned}
$$

4. Verify that the given function is a solution of the corresponding integral equations or integro-differential equations:

$$
\begin{aligned}
(a)\quad & u(x) = x - \int_0^x (x-t)u(t)dt, \quad u(x) = \sin x \\
(b)\quad & \int_0^x (x-t)^2 u(t)dt = x^3, \quad u(x) = 3 \\
(c)\quad & \int_0^x \sqrt{(x-t)}u(t)dt = x^{3/2}, \quad u(x) = 3/2.
\end{aligned}
$$

5. Reduce each of the Volterra integral equations to an equivalent initial value problem:

$$(a)\quad u(x) = x - \cos x + \int_0^x (x-t)u(t)dt$$

$$(b)\quad u(x) = x^4 + x^2 + 2\int_0^x (x-t)^2 u(t)dt$$

$$(c)\quad u(x) = x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)dt.$$

6. Derive an equivalent Volterra integral equation to each of the following initial value problems:

$$(a)\quad y'' + 5y' + 6y = 0, \quad y(0) = 1, y'(0) = 1$$

$$(b)\quad y'' + y = \sin x, \quad y(0) = 0, y'(0) = 0$$

$$(c)\quad y''' + 4y' = x, \quad y(0) = 0, y'(0) = 0, y''(0) = 1$$

$$(d)\quad \frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 2e^x, \quad y(0) = 2, y'(0) = 2, y''(0) = 1, y'''(0) = 1.$$

7. Derive the equivalent Fredholm integral equation for each of the following boundary value problems:

$$(a)\quad y'' + 4y = \sin x, 0 < x < 1, \quad y(0) = 0, y(1) = 0$$

$$(b)\quad y'' + 2xy = 1, 0 < x < 1, \quad y(0) = 0, y(1) = 0$$

$$(c)\quad y'' + y = x, 0 < x < 1, \quad y(0) = 1, y(1) = 0$$

$$(d)\quad y'' + y = x, 0 < x < 1, \quad y(0) = 1, y'(1) = 0.$$

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2 Volterra integral equations

2.1 Introduction

In the previous chapter, we have clearly defined the integral equations with some useful illustrations. This chapter deals with the Volterra integral equations and their solution techniques. The principal investigators of the theory of integral equations are Vito Volterra (1860–1940) and Ivar Fredholm (1866–1927), together with David Hilbert (1862–1943) and Erhard Schmidt (1876–1959). Volterra was the first to recognize the importance of the theory and study it systematically.

In this chapter, we shall be concerned with the nonhomogeneous Volterra integral equation of the second kind of the form

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \tag{2.1}$$

where $K(x,t)$ is the kernel of the integral equation, $f(x)$ a continuous function of x, and λ a parameter. Here, $f(x)$ and $K(x,t)$ are the given functions but $u(x)$ is an unknown function that needs to be determined. The limits of integral for the Volterra integral equations are functions of x. The nonhomogeneous Volterra integral equation of the first kind is defined as

$$\int_0^x K(x,t)u(t)dt = f(x) \tag{2.2}$$

We shall begin our study with this relatively simple, important class of integral equations in which many features of the general theory already appeared in the literatures. There are a host of solution techniques to deal with the Volterra integral equations. The Volterra integral equations of the second kind can be readily solved by using the Picard's process of successive approximations.

2.2 The method of successive approximations

In this method, we replace the unknown function $u(x)$ under the integral sign of the Volterra equation (2.1) by any selective real-valued continuous function $u_0(x)$,

called the zeroth approximation. This substitution will give the first approximation $u_1(x)$ by

$$u_1(x) = f(x) + \lambda \int_0^x K(x,t)u_0(t)dt \tag{2.3}$$

It is obvious that $u_1(x)$ is continuous if $f(x)$, $K(x,t)$, and $u_0(x)$ are continuous. The second approximation $u_2(x)$ can be obtained similarly by replacing $u_0(x)$ in equation (2.3) by $u_1(x)$ obtained above. And we find

$$u_2(x) = f(x) + \lambda \int_0^x K(x,t)u_1(t)dt \tag{2.4}$$

Continuing in this manner, we obtain an infinite sequence of functions

$$u_0(x), u_1(x), u_2(x), \ldots, u_n(x), \ldots$$

that satisfies the recurrence relation

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)dt \tag{2.5}$$

for $n = 1, 2, 3, \ldots$ and $u_0(x)$ is equivalent to any selected real-valued function. The most commonly selected function for $u_0(x)$ are 0, 1, and x. Thus, at the limit, the solution $u(x)$ of the equation (2.1) is obtained as

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{2.6}$$

so that the resulting solution $u(x)$ is independent of the choice of the zeroth approximation $u_0(x)$. This process of approximation is extremely simple. However, if we follow the Picard's successive approximation method, we need to set $u_0(x) = f(x)$, and determine $u_1(x)$ and other successive approximation as follows:

$$\begin{aligned}
u_1(x) &= f(x) + \lambda \int_0^x K(x,t)f(t)dt \\
u_2(x) &= f(x) + \lambda \int_0^x K(x,t)u_1(t)dt \\
&\cdots\cdots\cdots \\
u_{n-1}(x) &= f(x) + \lambda \int_0^x K(x,t)u_{n-2}(t)dt \\
u_n(x) &= f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)dt
\end{aligned} \tag{2.7}$$

The last equation is the recurrence relation. Consider

$$\begin{aligned} u_2(x) - u_1(x) &= \lambda \int_0^x K(x,t)[f(t) + \lambda \int_0^t K(t,\tau)f(\tau)d\tau]dt \\ &\quad - \lambda \int_0^x K(x,t)f(t)dt \\ &= \lambda^2 \int_0^x K(x,t) \int_0^t K(t,\tau)f(\tau)d\tau dt \\ &= \lambda^2 \psi_2(x) \end{aligned} \tag{2.8}$$

where

$$\psi_2(x) = \int_0^x K(x,t)dt \int_0^t K(t,\tau)f(\tau)d\tau \tag{2.9}$$

Thus, it can be easily observed from equation (2.8) that

$$u_n(x) = \sum_{m=0}^{n} \lambda^m \psi_m(x) \tag{2.10}$$

if $\psi_0(x) = f(x)$, and further that

$$\psi_m(x) = \int_0^x K(x,t)\psi_{m-1}(t)dt, \tag{2.11}$$

where $m = 1, 2, 3, \ldots$ and hence $\psi_1(x) = \int_0^x K(x,t)f(t)dt$.

The repeated integrals in equation (2.9) may be considered as a double integral over the triangular region indicated in Figure 2.1; thus interchanging the order of

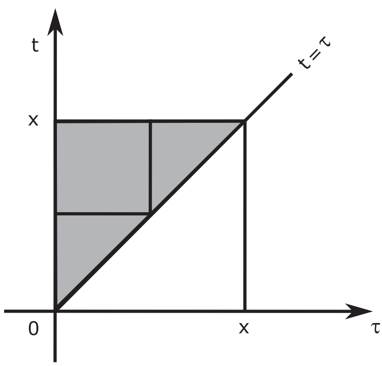

Figure 2.1: Double integration over the triangular region (shaded area).

integration, we obtain

$$
\begin{aligned}
\psi_2(x) &= \int_0^x f(\tau)d\tau \int_\tau^x K(x,t)K(t,\tau)dt \\
&= \int_0^x K_2(x,\tau)f(\tau)d\tau
\end{aligned}
$$

where $K_2(x,\tau) = \int_\tau^x K(x,t)K(t,\tau)dt$. Similarly, we find in general

$$
\psi_m(x) = \int_0^x K_m(x,\tau)f(\tau)d\tau, \quad m = 1,2,3,\ldots \tag{2.12}
$$

where the iterative kernels $K_1(x,t) \equiv K(x,t)$, $K_2(x,t)$, $K_3(x,t),\ldots$ are defined by the recurrence formula

$$
K_{m+1}(x,t) = \int_t^x K(x,\tau)K_m(\tau,t)d\tau, \quad m = 1,2,3,\ldots \tag{2.13}
$$

Thus, the solution for $u_n(x)$ can be written as

$$
u_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \psi_m(x) \tag{2.14}
$$

It is also plausible that we should be led to the solution of equation (2.1) by means of the sum if it exists, of the infinite series defined by equation (2.10). Thus, we have using equation (2.12)

$$
\begin{aligned}
u_n(x) &= f(x) + \sum_{m=1}^{n} \lambda^m \int_0^x K_m(x,\tau)f(\tau)d\tau \\
&= f(x) + \int_0^x \left\{ \sum_{m=1}^{n} \lambda^m K_m(x,\tau) \right\} f(\tau)d\tau; \qquad (2.15)
\end{aligned}
$$

hence it is also plausible that the solution of equation (2.1) will be given by as $n \to \infty$

$$
\begin{aligned}
\lim_{n\to\infty} u_n(x) &= u(x) \\
&= f(x) + \int_0^x \left\{ \sum_{m=1}^{n} \lambda^m K_m(x,\tau) \right\} f(\tau)d\tau \\
&= f(x) + \lambda \int_0^x H(x,\tau;\lambda)f(\tau)d\tau \qquad (2.16)
\end{aligned}
$$

where

$$H(x,\tau;\lambda) = \sum_{m=1}^{n} \lambda^m K_m(x,\tau) \tag{2.17}$$

is known as the resolvent kernel.

2.3 The method of Laplace transform

Volterra integral equations of convolution type such as

$$u(x) = f(x) + \lambda \int_0^x K(x-t)u(t)dt \tag{2.18}$$

where the kernel $K(x-t)$ is of convolution type, can very easily be solved using the Laplace transform method [1]. To begin the solution process, we first define the Laplace transform of $u(x)$

$$\mathcal{L}\{u(x)\} = \int_0^\infty e^{-sx}u(x)dx. \tag{2.19}$$

Using the Laplace transform of the convolution integral, we have

$$\mathcal{L}\left\{\int_0^x K(x-t)u(t)dt\right\} = \mathcal{L}\{K(x)\}\mathcal{L}\{u(x)\} \tag{2.20}$$

Thus, taking the Laplace transform of equation (2.18), we obtain

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{f(x)\} + \lambda\mathcal{L}\{K(x)\}\mathcal{L}\{u(x)\}$$

and the solution for $\mathcal{L}\{u(x)\}$ is given by

$$\mathcal{L}\{u(x)\} = \frac{\mathcal{L}\{f(x)\}}{1-\lambda\mathcal{L}\{K(x)\}},$$

and inverting this transform, we obtain

$$u(x) = \int_0^x \psi(x-t)f(t)dt \tag{2.21}$$

where it is assumed that $\mathcal{L}^{-1}\left\{\frac{1}{1-\lambda\mathcal{L}\{K(x)\}}\right\} = \psi(x)$. The expression (2.21) is the solution of the second kind Volterra integral equation of convolution type.

Example 2.1

Solve the following Volterra integral equation of the second kind of the convolution type using (a) the Laplace transform method and (b) successive approximation method

$$u(x) = f(x) + \lambda \int_0^x e^{x-t} u(t)dt \tag{2.22}$$

Solution

(a) Solution by Laplace transform method
Taking the Laplace transform of equation (2.22) and we obtain

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{f(x)\} + \lambda \mathcal{L}\{e^x\}\mathcal{L}\{u(x)\},$$

and solving for $\mathcal{L}\{u(x)\}$ yields

$$\mathcal{L}\{u(x)\} = \left(1 + \frac{\lambda}{s - 1 - \lambda}\right) \mathcal{L}\{f(x)\}.$$

The Laplace inverse of the above can be written immediately as

$$\begin{aligned} u(x) &= \int_0^x \{\delta(x-t) + \lambda e^{(1+\lambda)(x-t)}\} f(t)dt \\ &= f(x) + \lambda \int_0^x e^{(1+\lambda)(x-t)} f(t)dt \end{aligned} \tag{2.23}$$

where $\delta(x)$ is the Dirac delta function and we have used the integral property [7] to evaluate the integral. Because of the convolution type kernel, the result is amazingly simple.

(b) Solution by successive approximation
Let us assume that the zeroth approximation is

$$u_0(x) = 0 \tag{2.24}$$

Then the first approximation can be obtained as

$$u_1(x) = f(x) \tag{2.25}$$

Using this information in equation (2.22), the second approximation is given by

$$u_2(x) = f(x) + \lambda \int_0^x e^{x-t} f(t)dt \tag{2.26}$$

Proceeding in this manner, the third approximation can be obtained as

$$\begin{aligned}
u_3(x) &= f(x) + \lambda \int_0^x e^{x-t} u_2(t)dt \\
&= f(x) + \lambda \int_0^x e^{x-t} \left\{ f(t) + \lambda \int_0^t e^{t-\tau} f(\tau)d\tau \right\} dt \\
&= f(x) + \lambda \int_0^x e^{x-t} f(t)dt + \lambda^2 \int_0^x \int_0^t e^{x-\tau} f(\tau)d\tau dt \\
&= f(x) + \lambda \int_0^x e^{x-t} f(t)dt + \lambda^2 \int_0^x (x-t)e^{x-t} f(t)dt
\end{aligned}$$

In the double integration the order of integration is changed to obtain the final result.

In a similar manner, the fourth approximation $u_4(x)$ can be at once written as

$$\begin{aligned}
u_4(x) &= f(x) + \lambda \int_0^x e^{x-t} f(t)dt + \lambda^2 \int_0^x (x-t)e^{x-t} f(t)dt \\
&\quad + \lambda^3 \int_0^x \frac{(x-t)^2}{2!} e^{x-t} f(t)dt.
\end{aligned}$$

Thus, continuing in this manner, we obtain as $n \to \infty$

$$\begin{aligned}
u(x) &= \lim_{n\to\infty} u_n(x) \\
&= f(x) + \lambda \left\{ \int_0^x e^{x-t} \left(1 + \lambda(x-t) + \frac{1}{2!}\lambda^2(x-t)^2 + \cdots \right) f(t)dt \right\} \\
&= f(x) + \lambda \int_0^x e^{(x-t)}.e^{\lambda(x-t)} f(t)dt \\
&= f(x) + \lambda \int_0^x e^{(1+\lambda)(x-t)} f(t)dt \qquad (2.27)
\end{aligned}$$

which is the same as equation (2.23). Here, the resolvent kernel is $H(x,t;\lambda) = e^{(1+\lambda)(x-t)}$.

(c) Another method to determine the solution by the resolvent kernel

The procedure to determine the resolvent kernel is the following: Given that

$$u(x) = f(x) + \lambda \int_0^x e^{x-t} u(t)dt.$$

Here, the kernel is $K(x,t) = e^{x-t}$. The solution by the successive approximation is

$$u(x) = f(x) + \lambda \int_0^x H(x,t;\lambda) f(t)dt$$

where the resolvent kernel is given by

$$H(x,t;\lambda) = \sum_{n=0} \lambda^n K_{n+1}(x,t)$$

in which

$$K_{n+1}(x,t) = \int_t^x K(x,\tau)K_n(\tau)d\tau, \quad n = 1, 2, 3, \ldots$$

It is to be noted that $K_1(x,t) = K(x,t)$.

Thus, we obtain

$$\begin{aligned} K_2(x,t) &= \int_t^x e^{x-\tau}e^{\tau-t}d\tau \\ &= e^{x-t}\int_t^x d\tau \\ &= (x-t)e^{x-t} \end{aligned}$$

Similarly, proceeding in this manner, we obtain

$$\begin{aligned} K_3(x,t) &= \int_t^x (e^{x-\tau})(e^{\tau-t}(\tau-t))d\tau \\ &= e^{x-t}\frac{(x-t)^2}{2!} \\ K_4(x,t) &= e^{x-t}\frac{(x-t)^3}{3!} \\ \cdots\cdots\cdots & \quad \cdots\cdots\cdots \\ K_{n+1}(x,t) &= e^{x-t}\frac{(x-t)^n}{n!} \end{aligned}$$

Hence the resolvent kernel is

$$\begin{aligned} H(x,t;\lambda) &= \sum_{n=0} \lambda^n K_{n+1}(x,t) \\ &= e^{x-t}\sum_0^\infty \frac{(\lambda(x-t))^n}{n!} \\ &= e^{(1+\lambda)(x-t)} \end{aligned}$$

Once the resolvent kernel is known the solution is obvious.

Example 2.2

Solve the following linear Volterra integral equation

$$u(x) = x + \int_0^x (t - x)u(t)dt. \tag{2.28}$$

Solution

(a) Solution by Laplace transform method
The Laplace transform of equation (2.28) yields

$$\begin{aligned}\mathcal{L}\{u(x)\} &= \mathcal{L}\{x\} - \mathcal{L}\{x\}\mathcal{L}\{u(x)\} \\ &= \frac{1}{s^2} - \frac{1}{s^2}\mathcal{L}\{u(x)\}\end{aligned}$$

which reduces to $\mathcal{L}\{u(x)\} = \frac{1}{1+s^2}$ and its inverse solution is $u(x) = \sin x$. This is required solution.

(b) Solution by successive approximation
Let us set $u_0(x) = x$, then the first approximation is

$$\begin{aligned}u_1(x,t) &= x + \int_0^x (t - x)tdt \\ &= x - \frac{x^3}{3!}\end{aligned}$$

The second approximation can be calculated in the similar way that gives $u_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$. Proceeding in this way, we can obtain without much difficulty $u_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. The final solution is

$$\begin{aligned}u(x) &= \lim_{n\to\infty} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= \sin x\end{aligned}$$

This is the same as before.

2.4 The method of successive substitutions

The Volterra integral equation of the second kind is rewritten here for ready reference,

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \tag{2.29}$$

In this method, we substitute successively for $u(x)$ its value as given by equation (2.29). We find that

$$\begin{aligned}
u(x) &= f(x) + \lambda \int_0^x K(x,t) \left\{ f(t)\lambda \int_0^t K(t,t_1)u(t_1)dt_1 \right\} dt \\
&= f(x) + \lambda \int_0^x K(x,t)f(t)dt + \lambda^2 \int_0^x K(x,t) \int_0^t K(t,t_1)u(t_1)dt_1 dt \\
&= f(x) + \lambda \int_0^x K(x,t)f(t)dt + \lambda^2 \int_0^x K(x,t) \int_0^t K(t,t_1)f(t_1)dt_1 dt \\
&\quad + \cdots \\
&\quad + \lambda^n \int_0^x K(x,t) \int_0^t K(t,t_1) \cdots \\
&\quad \times \int_0^{t_{n-2}} K(t_{n-2},t_{n-1})f(t_{n-1})dt_{n-1} \cdots dt_1 dt + R_{n+1}(x)
\end{aligned}$$

where

$$R_{n+1} = \lambda^{n+1} \int_0^x K(x,t) \int_0^t K(t,t_1) \cdots \int_0^{t_{n-1}} K(t_{n-1},t_n)u(t_n)dt_n \cdots dt_1 dt$$

is the remainder after n terms. It can be easily shown that (see Ref. [8]) that $\lim_{n\to\infty} R_{n+1} = 0$. Accordingly, the general series for $u(x)$ can be written as

$$\begin{aligned}
u(x) &= f(x) + \lambda \int_0^x K(x,t)f(t)dt \\
&\quad + \lambda^2 \int_0^x \int_0^t K(x,t)K(t,t_1)f(t_1)dt_1 dt \\
&\quad + \lambda^3 \int_0^x \int_0^t \int_0^{t_1} K(x,t)K(t,t_1)K(t_1,t_2)f(t_2)dt_2 dt_1 dt \\
&\quad + \cdots
\end{aligned} \tag{2.30}$$

It is to be noted here that in this method the unknown function $u(x)$ is substituted by the given function $f(x)$ that makes the evaluation of the multiple integrals easily computable.

Theorem 2.1

If

(*a*) $u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt$, *a is a constant.*
(*b*) $K(x,t)$ *is real and continuous in the rectangle* R, *for which* $a \le x \le b$, $a \le t \le b$, $|K(x,t)| \le M$ *in* R, $K(x,t) \ne 0$.
(*c*) $f(x) \ne 0$, *is real and continuous in* $I : a \le x \le b$.
(*d*) λ, *a constant.*

then the equation (2.29) *has one and only one continuous solution* $u(x)$ *in* I, *and this solution is given by the absolutely and uniformly convergent series* (2.30).

The results of this theorem hold without change for the equation

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt, \quad \text{for } \lambda = 1.$$

Example 2.3

Solve the following linear Volterra equation

$$u(x) = \cos x - x - 2 + \int_0^x (t-x)u(t)dt. \tag{2.31}$$

Solution

(a) The Laplace transform method
Take the Laplace transform of equation (2.31), and we obtain

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{\cos x\} - \mathcal{L}\{x\} - 2\mathcal{L}\{1\} + \mathcal{L}\{\int_0^x (t-x)u(t)dt\}$$

$$= \frac{s}{1+s^2} - \frac{1}{s^2} - \frac{2}{s} - \frac{1}{s^2}\mathcal{L}\{u(x)\}$$

Simplification of the above equation yields

$$\mathcal{L}\{u(x)\} = \frac{s^3}{(1+s^2)^2} - \frac{1}{1+s^2} - \frac{2s}{1+s^2}$$

The Laplace inverse of $\frac{s^3}{(1+s^2)^2}$ needs the knowledge of partial fraction and the convolution integral. The result can be computed as

$$\mathcal{L}^{-1}\left\{\frac{s^3}{(1+s^2)^2}\right\} = \cos x - \frac{x}{2}\sin x$$

and hence the solution is

$$u(x) = -\cos x - \sin x - \frac{x}{2}\sin x.$$

This result can be verified easily by putting the solution in the integral equation (2.31). The problem in here is to calculate $\int_0^x (t-x)(-\cos x - \sin x - t/2\sin x)dt$ which is equal to $x - 2\cos x - \sin x - x/2\sin x + 2$, and simply add to $\cos x - x - 2$. This reduces to the desired result.

Example 2.4

Solve the integral $u(x) = 1 + \int_0^x u(t)dt$.

Solution

(a) By the Laplace transform method, we have $\mathcal{L}\{u(x)\} = \frac{1}{s-1}$. The inverse is simply $u(x) = e^x$ which is the required solution.
(b) By the method of successive substitution, we obtain

$$
\begin{aligned}
u(x) &= 1 \int_0^x dt + \int_0^x \int_0^x dt^2 + \int_0^x \int_0^x \int_0^x dt^3 + \cdots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
&= e^x.
\end{aligned}
$$

These two solutions are identical. Hence it is the solution.

2.5 The Adomian decomposition method

The Adomian decomposition method appears to work for linear and nonlinear differential equations, integral equations, integro-differential equations. The method was introduced by Adomian in early 1990 in his books [1] and [2] and other related research papers [3] and [4]. The method essentially is a power series method similar to the perturbation technique. We shall demonstrate the method by expressing $u(x)$ in the form of a series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{2.32}$$

with $u_0(x)$ as the term outside the integral sign.

The integral equation is

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \tag{2.33}$$

and hence

$$u_0(x) = f(x) \tag{2.34}$$

Substituting equation (2.32) into equation (2.33) yields

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t) \left\{ \sum_{n=0}^{\infty} u_n(t) \right\} dt \tag{2.35}$$

The components $u_0(x), u_1(x), u_2(x), \ldots u_n(x) \ldots$ of the unknown function $u(x)$ can be completely determined in a recurrence manner if we set

$$\begin{aligned}
u_0(x) &= f(x) \\
u_1(x) &= \lambda \int_0^x K(x,t)u_0(t)dt \\
u_2(x) &= \lambda \int_0^x K(x,t)u_1(t)dt \\
\cdots\cdots &= \cdots\cdots \\
u_n(x) &= \lambda \int_0^x K(x,t)u_{n-1}(t)dt
\end{aligned} \tag{2.36}$$

and so on. This set of equations (2.36) can be written in compact recurrence scheme as

$$\begin{aligned}
u_0(x) &= f(x) \\
u_{n+1}(x) &= \lambda \int_0^x K(x,t)u_n(t)dt, \quad n \geq 0
\end{aligned} \tag{2.37}$$

It is worth noting here that it may not be possible to integrate the kernel for many components. In that case, we truncate the series at a certain point to approximate the function $u(x)$. There is another point that needs to be addressed; that is the convergence of the solution of the infinite series. This problem was addressed by many previous workers in this area (see Refs. [5] and [6]). So, it will not be repeated here. We shall demonstrate the technique with some examples.

Example 2.5

Solve the Volterra integral equation $u(x) = x + \int_0^x (t-x)u(t)dt$ by the decomposition method.

Solution

Consider the solution in the series form $u(x) = \sum_{n=0}^{\infty} u_n(x)$. Then substituting this series into the given equation, we have

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x (t-x) \sum_{n=0}^{\infty} u_n(t)dt.$$

Now decomposing the different terms in the following manner, we get a set of solutions

$$\begin{aligned}
u_0(x) &= x \\
u_1(x) &= \int_0^x (t-x)u_0(t)dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^x (t-x)tdt \\
&= -\frac{x^3}{3!} \\
u_2(x) &= \int_0^x (t-x)u_1(t)dt \\
&= \int_0^x (t-x)(-\frac{t^3}{3!})dt \\
&= \frac{x^5}{5!}
\end{aligned}
$$

Continuing in this way we obtain a series

$$
u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sin x
$$

which is the closed-form solution of the given integral equation. By the ratio test where $u_n(x) = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, it can be easily shown that $|x| < \infty$ that means the series is convergent for all values of x in a finite domain. By taking the Laplace transform of the integral equation, it is easy to see that $\mathcal{L}\{u(x)\} = \frac{1}{s^2+1}$ and so $u(x) = \sin x$.

Example 2.6

Solve the integral equation $u(x) = f(x) + \lambda \int_0^x e^{x-t}u(t)dt$ by the decomposition method.

Solution

Let us consider $u(x) = \sum_{n=0}^{\infty} u_n(x)$ is the solution of the equation. Hence substituting into the equation we have

$$
\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x e^{x-t} \sum_{n=0}^{\infty} u_n(t)dt.
$$

Hence equating the like terms we have

$$
\begin{aligned}
u_0(x) &= f(x) \\
u_1(x) &= \lambda \int_0^x e^{x-t} f(t)dt \\
u_2(x) &= \lambda \int_0^x e^{x-t} u_1(t)dt
\end{aligned}
$$

$$= \lambda \int_0^x e^{x-t} \left\{ \lambda \int_0^t e^{t-t_1} f(t_1) dt_1 \right\} dt$$
$$= \lambda^2 \int_0^x f(t_1) dt_1 \int_{t_1}^x e^{x-t} e^{t-t_1} dt$$
$$= \lambda^2 \int_0^x e^{x-t_1} (x - t_1) f(t_1) dt_1$$
$$= \lambda^2 \int_0^x (x - t) e^{x-t} f(t) dt.$$

Note here that $u_2(x) = \lambda^2 \int_0^x \int_0^x e^{x-t} f(t) dt dt = \lambda^2 \int_0^x (x - t) e^{x-t} f(t) dt$. Similarly, we have

$$u_3(x) = \lambda \int_0^x e^{x-t} u_2(t) dt$$
$$= \lambda^3 \int_0^x \frac{(x-t)^2}{2!} e^{x-t} f(t) dt.$$

Thus, the decomposition series becomes

$$u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$
$$= f(x) + \lambda \int_0^x \left\{ 1 + \lambda(x - t) + \lambda^2 \frac{(x-t)^2}{2!} + \cdots \right\} e^{x-t} f(t) dt$$
$$= f(x) + \lambda \int_0^x e^{\lambda(x-t)} e^{x-t} f(t) dt$$
$$= f(x) + \lambda \int_0^x e^{(1+\lambda)(x-t)} f(t) dt.$$

2.6 The series solution method

We shall introduce a practical method to handle the Volterra integral equation

$$u(x) = f(x) + \lambda \int_0^x K(x, t) u(t) dt.$$

In the series solution method we shall follow a parallel approach known as the Frobenius series solution usually applied in solving the ordinary differential equation around an ordinary point (see Ref. [9]). The method is applicable provided that $u(x)$ is an analytic function, i.e. $u(x)$ has a Taylor's expansion around $x = 0$. Accordingly, $u(x)$ can be expressed by a series expansion given by

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2.38}$$

where the coefficients a and x are constants that are required to be determined. Substitution of equation (2.38) into the above Volterra equation yields

$$\sum_{n=0}^{\infty} a_n x^n = f(x) + \lambda \int_0^x K(x,t) \sum_{n=0}^{\infty} a_n t^n dt \tag{2.39}$$

so that using a few terms of the expansion in both sides, we find

$$\begin{aligned}
& a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\
&= f(x) + \lambda \int_0^x K(x,t) a_0 dt + \lambda \int_0^x K(x,t) a_1 t dt \\
&\quad + \lambda \int_0^x K(x,t) a_2 t^2 dt + \cdots + \lambda \int_0^x K(x,t) a_n t^n dt + \cdots
\end{aligned} \tag{2.40}$$

In view of equation (2.40), the integral equation will be reduced to several traditional integrals, with defined integrals having terms of the form t^n, $n \geq 0$ only. We then write the Taylor's expansions for $f(x)$ and evaluate the first few integrals in equation (2.40). Having performed the integration, we equate the coefficients of like powers of x in both sides of equation (2.40). This will lead to a complete determination of the unknown coefficients $a_0, a_1, a_2, \ldots a_n \ldots$. Consequently, substituting these coefficients $a_n, n \geq 0$, which are determined in equation (2.40), produces the solution in a series form. We will illustrate the series solution method by a simple example.

Example 2.7

Obtain the solution of the Volterra equation $u(x) = 1 + 2\sin x - \int_0^x u(t)dt$ using the series method.

Solution

We assume the solution in the series form $u(x) = \sum_{n=0}^{\infty} a_n x^n$. Hence substituting the series into the equation and the Taylor's series of $\sin x$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= 1 + 2\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} - \int_0^x \sum_{n=0}^{\infty} a_n t^n dt \\
&= 1 + 2\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{(n+1)!}
\end{aligned}$$

Comparing the coefficients of the same power of x gives the following set of values:

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 2 - a_0 \\
a_2 &= -\frac{a_1}{2} \\
a_3 &= -\frac{2}{3!} - \frac{a_2}{3} \\
a_4 &= -\frac{a_3}{4!}
\end{aligned}$$

and so on. Thus, the values of the coefficients can be computed to be $a_0 = 1$, $a_1 = 1, a_2 = -\frac{1}{2}, a_3 = -\frac{1}{3!}, a_4 = \frac{1}{4!}, \ldots$. Hence the solution is given by

$$\begin{aligned}
u(x) &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= \cos x + \sin x
\end{aligned}$$

By the Laplace transform method it can be easily verified that

$$\mathcal{L}\{u(x)\} = \frac{1}{s} + \frac{2}{s^2+1} - \frac{1}{s^2}\mathcal{L}\{u(x)\}$$

and hence simplifying we get

$$\mathcal{L}\{u(x)\} = \frac{s+1}{s^2+1}$$

the inverse of which simply is $u(x) = \cos x + \sin x$. These two solutions are identical confirming that this is the required solution.

2.7 Volterra equation of the first kind

The relation between Volterra integral equations of the first and the second kind can be established in the following manner. The first kind Volterra equation is usually written as

$$\int_0^x K(x,t)u(t)dt = f(x) \tag{2.41}$$

If the derivatives $\frac{df}{dx} = f'(x)$, $\frac{\partial K}{\partial x} = K_x(x,t)$, and $\frac{\partial K}{\partial t} = K_t(x,t)$ exist and are continuous, the equation can be reduced to one of the second kind in two ways. The first and the simple way is to differentiate both sides of equation (2.41) with respect to x and we obtain the equation by using the Leibnitz rule

$$K(x,x)u(x) + \int_0^x k_x(x,t)u(t)dt = f'(x) \tag{2.42}$$

If $K(x,x) \neq 0$, then dividing throughout by this we obtain

$$K(x,x)u(x) + \int_0^x k_x(x,t)u(t)dt = f'(x) \tag{2.43}$$

$$u(x) + \int_0^x \frac{k_x(x,t)}{K(x,x)}u(t)dt = \frac{f'(x)}{K(x,x)} \tag{2.44}$$

and the reduction is accomplished. Thus, we can use the method already given above.

The second way to obtain the second kind Volterra integral equation from the first kind is by using integration by parts, if we set

$$\int_0^x u(t)dt = \phi(x)$$

$$\text{or equivalently,} \quad \int_0^t u(\xi)d\xi = \phi(t) \tag{2.45}$$

We obtain the equation by integration by parts,

$$\left[K(x,t)\int_0^t u(\xi)d\xi\right]_{t=0}^x - \int_0^x K_t(x,t)\left(\int_0^t u(\xi)d\xi\right)dt = f(x)$$

which reduces to

$$[K(x,t)\phi(t)]_{t=0}^x - \int_0^x K_t(x,t)\phi(t)dt = f(x)$$

and finally we get

$$K(x,x)\phi(x) - K(x,0)\phi(0) - \int_0^x K_t(x,t)\phi(t)dt = f(x) \tag{2.46}$$

It is obvious that $\phi(0) = 0$, and dividing out by $K(x,x)$ we have

$$\begin{aligned}\phi(x) &= \left\{\frac{f(x)}{K(x,x)}\right\} + \int_0^x \left\{\frac{K_t(x,t)}{K(x,x)}\right\}\phi(t)dt \\ &= F(x) + \int_0^x G(x,t)\phi(t)dt \end{aligned} \tag{2.47}$$

where $F(x) = \frac{f(x)}{K(x,x)}$ and $G(x,t) = \frac{K_t(x,t)}{K(x,x)}$.

For this second process it is apparently not necessary for $f(x)$ to be differentiable. However, the function $u(x)$ must finally be calculated by differentiating the function $\phi(x)$ given by the formula

$$\phi(x) = \left\{\frac{f(x)}{K(x,x)}\right\} + \int_0^x H(x,t;1)\left\{\frac{f(t)}{K(t,t)}\right\}dt \tag{2.48}$$

where $H(x,t:1)$ is the resolvent kernel corresponding to $\frac{K_t(x,t)}{K(x,x)}$. To do this $f(x)$ must be differentiable.

Remark

If $K(x,x)$ vanishes at some point of the basic interval $(0 \le x \le b)$, for instance at $x = 0$, then the equations (2.44) and (2.47) will have a peculiar character, essentially different from that of equation of the second kind. These equations are called by Picard the equation of the third kind. However, if $K(x,x)$ vanishes identically, it is sometimes possible to obtain an equation of the second kind by previous transformations.

A special case of Volterra integral equation

In the second kind Volterra equation if the kernel $K(x,t)$ is assumed to be $K(x,t) = \frac{A(x)}{A(t)}$ such that the equation takes the form

$$u(x) = f(x) + \lambda \int_0^x \frac{A(x)}{A(t)} u(t)dt \tag{2.49}$$

and upon dividing throughout by $A(x)$ yields

$$\left\{\frac{u(x)}{A(x)}\right\} = \left\{\frac{f(x)}{A(x)}\right\} + \lambda \int_0^x \left\{\frac{u(t)}{A(t)}\right\} dt \tag{2.50}$$

Now define $\frac{u(x)}{A(x)} = u_1(x)$ and $\frac{f(x)}{A(x)} = f_1(x)$ and equation (2.50) can be written as

$$u_1(x) = f_1(x) + \lambda \int_0^x u_1(t)dt \tag{2.51}$$

Assuming that $u_2(x) = \int_0^x u_1(t)dt$, equation (2.51) can be reduced to an ordinary differential equation

$$\frac{du_2}{dx} - \lambda u_2 = f_1(x) \tag{2.52}$$

the general solution of which can be obtained as

$$u_2(x) = \int_0^x e^{\lambda(x-t)} f_1(t)dt + C_1 \tag{2.53}$$

Using the initial condition $u_2(0) = 0$ at $x = 0$, the equation (2.53) reduces to

$$u_2(x) = \int_0^x e^{\lambda(x-t)} f_1(t)dt.$$

But $u_1(x) = \frac{du_2}{dx}$ and so the above equation can be reduced to an integral equation in terms of u_1 by differentiating according to the Leibnitz rule to yield

$$u_1(x) = \lambda \int_0^x e^{\lambda(x-t)} f_1(t)dt + f_1(x) \tag{2.54}$$

Hence the solution to the original problem can be obtained multiplying throughout by $A(x)$

$$u(x) = f(x) + \lambda \int_0^x e^{\lambda(x-t)} f(t)dt \tag{2.55}$$

Obviously, this formula can also be obtained by the previous method of successive approximation.

2.8 Integral equations of the Faltung type

In this section, we shall be concerned with another type of Volterra integral equation of the second kind. If the kernel $K(x,t) \equiv K(x-t)$ then the second kind and the first kind will take the following forms, respectively,

$$u(x) = f(x) + \lambda \int_0^x K(x-t)u(t)dt \tag{2.56}$$

$$f(x) = \int_0^x K(x-t)u(t)dt \tag{2.57}$$

Equations (2.56) and (2.57) are very important special classes of Volterra integral equations, which Volterra called the equations of the closed cycle because the operator

$$V_x\{u(x)\} = \int_{-\infty}^x K(x,t)u(t)dt$$

carries any periodic function $u(t)$ with arbitrary period T into another periodic function with the same period T, if and only if $K(x,t) = K(x-t)$. Today, they are usually called equations of the Faltung type because the operation

$$\begin{aligned} u * v &= \int_0^x u(x-t)v(t)dt \\ &= \int_0^x u(t)v(x-t)dt \end{aligned} \tag{2.58}$$

is generally called Faltung (convolution) of the two functions u and v. In fact, setting $x - t = \tau$ such that $dt = -d\tau$, we obtain

$$u * v = \int_0^x u(\tau)v(x-\tau)d\tau = v * u$$

Finally, we notice that it is often convenient to set $u * u = u*^2; u * u *^2 = u*^3$ and so on. Using the Faltung sign, the integral equations (2.56) and (2.57) can be written, respectively, as

$$u(x) = f(x) + \lambda K(x) * u(x) \tag{2.59}$$

$$f(x) = K(x) * u(x) \tag{2.60}$$

The main device in dealing with such equations is the Laplace transformation

$$\mathcal{L}\{u(x)\} = \int_0^\infty e^{-sx} u(x)dx \tag{2.61}$$

because, under some restrictions, this operator transforms the convolution into an ordinary product

$$\mathcal{L}\{K(x) * u(x)\} = \mathcal{L}\{K(x)\}\mathcal{L}\{u(x)\}. \tag{2.62}$$

In this way, the integral equation can be transformed into an algebraic equation, and then solving $\mathcal{L}\{u(x)\}$, and inverting the transform we obtain the desired solution.

Example 2.8

Solve the convolution type Volterra integral equation

$$u(x) = x \int_0^x (t - x)u(t)dt.$$

Solution

By taking the Laplace transform of the given equation, we have

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{x\} - \mathcal{L}\{x\}\mathcal{L}\{u(x)\}$$

which reduces to

$$\mathcal{L}\{u(x)\} = \frac{1}{s^2 + 1}.$$

The inverse is given by

$$u(x) = \sin x$$

which is the required solution.

Example 2.9

Solve the following Abel's problem of the Faltung type

$$\frac{\pi x}{2} = \int_0^x \frac{1}{\sqrt{x - t}} u(t)dt.$$

Solution

Taking the Laplace transform of the given equation yields

$$\mathcal{L}\left\{\frac{\pi x}{2}\right\} = \mathcal{L}\left\{\int_0^x \frac{1}{\sqrt{x-t}}u(t)dt\right\},$$

which simply reduces to

$$\mathcal{L}\{u(x)\} = \frac{\sqrt{\pi}}{2}\frac{1}{s^{3/2}}.$$

Therefore, the inverse transform simply gives

$$u(x) = \sqrt{x}.$$

It is to be noted here that the recurrence relation of Gamma function is $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Example 2.10

Solve the Abel's problem

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt.$$

Solution

Taking the Laplace transform of the given equation yields

$$\mathcal{L}\{f(x)\} = \mathcal{L}\left\{\frac{1}{\sqrt{x}}\mathcal{L}\{u(x)\}\right\}.$$

This transformed equation reduces to

$$\mathcal{L}\{u(x)\} = \sqrt{\frac{s}{\pi}}\mathcal{L}\{f(x)\}.$$

Inversion of this problem is performed as follows:

$$\begin{aligned}
u(x) &= \frac{1}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\frac{s\mathcal{L}\{f(x)\}}{\sqrt{s}}\right\} \\
&= \frac{1}{\sqrt{\pi}}\frac{d}{dx}\mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{f(x)\}}{\sqrt{s}}\right\} \\
&= \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)dt}{\sqrt{x-t}}
\end{aligned}$$

which is the desired solution of the Abel equation.

Remark

It is clear that the Leibnitz rule is not applicable to the above solution because the integrand is discontinuous at the interval of integration. To determine the solution in another form, we first integrate because the integrand is the known function. Put $x - t = \tau^2$, and with this substitution, it is obvious that

$$\int_0^x \frac{f(t)}{\sqrt{x-t}} = 2\int_0^{\sqrt{x}} f(x-\tau^2)d\tau.$$

Thus, we obtain

$$u(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^{\sqrt{x}} f(x-\tau^2)d\tau.$$

Thus, using this information, the solution for $u(x)$ can be written as

$$u(x) = \frac{f(0)}{\sqrt{x}} + \int_0^x \frac{f'(t)}{\sqrt{x-t}}dt.$$

Example 2.11

Find the solution of the integral equation of Faltung type

$$u(x) = 2x^2 + \int_0^x \sin 4t u(x-t)dt.$$

Solution

Taking the Laplace transform of both sides,

$$\mathcal{L}\{u(x)\} = 2\mathcal{L}\{\sin 4t\}\mathcal{L}\{u(x)\},$$

after reduction, we obtain

$$\mathcal{L}\{u(x)\} = \frac{4(s^2+16)}{s^3(s^2+12)}.$$

The inverse Laplace transform is obtained by partial fraction

$$\begin{aligned} u(x) &= \mathcal{L}^{-1}\left\{\frac{4(s^2+16)}{s^3(s^2+12)}\right\} \\ &= \mathcal{L}^{-1}\left\{-\frac{1}{9s} + \frac{16}{3s^3} + \frac{s}{9(s^2+12)}\right\} \\ &= -\frac{1}{9} + \frac{8x^2}{3} + \frac{1}{9}\cos(2\sqrt{3}x) \end{aligned}$$

which is the desired solution.

2.9 Volterra integral equation and linear differential equations

There is a fundamental relationship between Volterra integral equations and ordinary differential equations. In fact, the solution of any differential equation of the type

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = F(x) \tag{2.63}$$

with continuous coefficients, together with the initial conditions

$$y(0) = c_0, y'(0) = c_1, y''(0) = c_2, \ldots, y^{(n-1)}(0) = c_{n-1} \tag{2.64}$$

can be reduced to the solution of a certain Volterra integral equation of the second kind

$$u(x) + \int_0^x K(x,t)u(t)dt = f(x) \tag{2.65}$$

In order to achieve this, let us make the transformation

$$\frac{d^n y}{dx^n} = u(x) \tag{2.66}$$

Hence integrating with respect to x from 0 to x

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_0^x u(t)dt + c_{n-1}$$

Thus, the successive integrals are

$$\begin{aligned}
\frac{d^{n-2}y}{dx^{n-2}} &= \int_0^x \int_0^x u(t)dtdt + c_{n-1}x + c_{n-2} \\
\frac{d^{n-3}y}{dx^{n-3}} &= \int_0^x \int_0^x \int_0^x u(t)dtdtdt + c_{n-1}\frac{x^2}{2!} + c_{n-2}x + c_{n-3} \\
\cdots\cdots &= \cdots\cdots
\end{aligned}$$

Proceeding in this manner we obtain

$$\begin{aligned}
y(x) &= \underbrace{\int_0^x \int_0^x \int_0^x \cdots \int_0^x}_{n\text{-fold integration}} u(t)dtdtdt\cdots dt \\
&\quad + c_{n-1}\frac{x^{n-1}}{(n-1)!} + c_{n-2}\frac{x^{n-2}}{(n-2)!} + \cdots + c_1x + c_0 \\
&= \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}u(t)dt + c_{n-1}\frac{x^{n-1}}{(n-1)!} + c_{n-2}\frac{x^{n-2}}{(n-2)!} + \cdots + c_1x + c_0
\end{aligned} \tag{2.67}$$

Returning to the differential equation (2.63), we see that it can be written as

$$
\begin{aligned}
&u(x)+\int_0^x \left\{a_1(x)+a_2(x)(x-t)+a_3(x)\frac{(x-t)^2}{2!}+\cdots+a_n(x)\frac{(x-t)^{n-1}}{(n-1)!}\right\}u(t)dt\\
&\quad = F(x)-c_{n-1}a_1(x)-(c_{n-1}+c_{n-2})a_2(x)-\cdots\\
&\qquad -(c_{n-1}\frac{x^{n-1}}{(n-1)!}+\cdots+c_1x+c_0)a_n(x)
\end{aligned}
$$

and this reduces to

$$u(x)+\int_0^x K(x,t)u(t)dt=f(x)$$

where

$$K(x,t)=\sum_{\nu=1}^{n}a_\nu(x)\frac{(x-t)^{\nu-1}}{(\nu-1)!} \tag{2.68}$$

and

$$
\begin{aligned}
f(x) &= F(x)-c_{n-1}a_1(x)-(c_{n-1}+c_{n-2})a_2(x)-\cdots\\
&\quad -\left(c_{n-1}\frac{x^{n-1}}{(n-1)!}+\cdots+c_1x+c_0\right)a_n(x).
\end{aligned} \tag{2.69}
$$

Conversely, the solution, i.e. equation (2.65) with K and f given by equations (2.68) and (2.69) and substituting values for $u(x)$ in the last equation of equation (2.67), we obtain the (unique) solution of equation (2.63) which satisfies the initial conditions (equation (2.64)). If the leading coefficient in equation (2.63) is not unity but $a_0(x)$, equation (2.65) becomes

$$a_0(x)u(x)+\int_0^x K(x,t)u(t)dt=f(x) \tag{2.70}$$

where K and f are still given by equations (2.68) and (2.69), respectively.

Remark

If $a_0(x)\neq 0$ in the interval considered, nothing is changed; however, if $a_0(x)$ vanishes at some point, we see that an equation of the type (2.70) is equivalent to a singular differential equation, at least, when $K(x,t)$ is a polynomial in t.

Example 2.12

Reduce the initial value problem

$$y''(x)+4y(x)=\sin x;\quad \text{at } x=0, y(0)=0, y'(0)=0$$

to Volterra integral equation of the second kind and then find its solution,

Solution

Volterra equation can be obtained in the following manner:

Let us consider

$$\begin{aligned}
y''(x) &= u(x) \\
y'(x) &= \int_0^x u(t)dt \\
y(x) &= \int_0^x \int_0^x u(t)dt^2 \\
&= \int_0^x (x-t)u(t)dt
\end{aligned}$$

Then the given ODE becomes

$$u(x) + 4\int_0^x (x-t)u(t)dt = \sin x$$

which is the Volterra integral equation. The solution can be obtained using the Laplace transform method and we obtain

$$\mathcal{L}\{u(x)\} + 4\mathcal{L}\{x\}\mathcal{L}\{u(x)\} = \frac{1}{s^2+1}$$

which reduces to

$$\mathcal{L}\{u(x)\} = \frac{s^2}{(s^2+1)(s^2+4)}.$$

By partial fraction, the inverse is obtained as

$$u(x) = -\frac{1}{3}\sin x + \frac{2}{3}\sin 2x.$$

Therefore, the solution for y results in

$$\begin{aligned}
y(x) &= \int_0^x (x-t)u(t)dt \\
&= \int_0^x (x-t)\left\{-\frac{1}{3}\sin t + \frac{2}{3}\sin 2t\right\} dt \\
&= \frac{1}{3}\left\{\sin x - \frac{1}{2}\sin 2x\right\}
\end{aligned}$$

Note that this solution agrees with that obtained from the differential equation.

2.10 Exercises

1. Solve the following Volterra integral equations by using decomposition method or modified decomposition method:

$$(a)\quad u(x) = 1 + x - x^2 + \int_0^x u(t)dt$$

$$(b)\quad u(x) = 1 + x + \int_0^x (x-t)u(t)dt$$

$$(c)\quad u(x) = 1 + \frac{x^2}{2!} - \int_0^x (x-t)u(t)dt$$

$$(d)\quad u(x) = \sec^2 x + (1 - e^{\tan x})x + x\int_0^x e^{\tan t}u(t)dt, \quad x < \pi/2$$

$$(e)\quad u(x) = x^3 - x^5 + 5\int_0^x tu(t)dt.$$

2. Solve the following Volterra integral equations by using series solution method:

$$(a)\quad u(x) = 2x + 2x^2 - x^3 + \int_0^x u(t)dt$$

$$(b)\quad u(x) = -1 - \int_0^x u(t)dt$$

$$(c)\quad u(x) = 1 - x - \int_0^x (x-t)u(t)dt$$

$$(d)\quad u(x) = x\cos x + \int_0^x tu(t)dt.$$

3. Solve the following Volterra integral equations of the first kind:

$$(a)\quad xe^{-x} = \int_0^x e^{t-x}u(t)dt$$

$$(b)\quad 5x^2 + x^3 = \int_0^x (5 + 3x - 3t)u(t)dt$$

$$(c)\quad 2\cosh x - \sinh x - (2-x) = \int_0^x (2 - x + t)u(t)dt$$

$$(d)\quad \tan x - \ln(\cos x) = \int_0^x (1 + x - t)u(t)dt, \quad x < \pi/2.$$

4. Solve the linear integral equation $u(x) = 1 + \int_0^x (t-x)u(t)dt$.
5. Solve the linear integral equation $u(x) = 9 + 6x + \int_0^x (6x - 6t + 5)u(t)dt$.

6. Solve the linear integral equation $u(x) = \cos x - x - 2 + \int_0^x (t-x)u(t)dt$.
7. Using the method of successive approximations find five successive approximations in the solution of exercises 1, 2, 3 after choosing $u_0(x) = 0$.
8. Show that

$$u(x) = a + bx + \int_0^x \{c + d(x-t)\}u(t)dt$$

where a, b, c, d are arbitrary constants, has for solution

$$u(x) = \alpha e^{\lambda x} + \beta e^{\nu x},$$

where $\alpha, \beta, \lambda, \nu$ depend upon a, b, c, d.

9. Solve the linear integral equation $u(x) = \frac{3e^x}{2} - \frac{xe^x}{2} - \frac{1}{2} + \frac{1}{2}\int_0^x tu(t)dt$.
10. The mathematical model of an electric circuit that contains the elements L, R, and C with an applied e.m.f. $E_0 \sin \omega t$ is given by the differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E_0 \sin \omega t,$$

where Q is the charge and $I = \frac{dQ}{dt}$ is the current flowing through the circuit. Given that $L = 0.5$, $R = 6$, $C = 0.02$, $E_0 = 24$ and $\omega = 10$ with the initial conditions $Q(0) = 0$ and $I(0) = 0$ at $t = 0$, find the charge $Q(t)$ for $t = 0$ to 10 s with a step size $h = 1$ s. Compare the numerical result with that of the analytical solution. (Hint: Express the given equation as a pair of first-order equations and then use the fourth-order Runge–Kutta method.)

11. The motion of a compound spring system is given by the solution of the pair of simultaneous equations

$$m_1\frac{d^2y_1}{dt^2} = -k_1y_1 - k_2(y_1 - y_2)$$
$$m_2\frac{d^2y_2}{dt^2} = k_2(y_1 - y_2)$$

where y_1 and y_2 are the displacements of the two masses from their equilibrium positions. The initial conditions are

$$y_1(0) = \alpha, \quad y_1'(0) = \beta, \quad y_2(0) = \gamma, \quad y_2'(0) = \delta.$$

Express as a set of first-order equations and then determine the numerical solutions using the fourth-order Runge–Kutta method.
(Hint: Use some suitable values of the parameters to calculate the displacements.)

12. For a resonant spring system with a periodic forcing function, the differential equation is

$$\frac{d^2y}{dt^2} + 9y = 4\cos 3t, \quad y(0) = y'(0) = 0.$$

Determine the displacement at $t = 0.1$ through $t = 0.8$ with $h = 0.1$ by using the Runge–Kutta method. Compare to the analytical solution $\frac{2}{3}t \sin 3t$.

13. In Exercise 1, if the applied voltage is 15 volts and the circuit elements are $R = 5$ ohms, $C = 1000$ microfarads, $h = 50$ millihenries, determine how the current varies with time between 0 and 0.2 s with $\Delta t = 0.005$ s. Compare this computation with the analytical solution.

14. In the theory of beams it is shown that the radius of curvature at any point is proportional to the bending moment

$$EI = \frac{y''}{[1 + (y')^2]^{3/2}} = M(x)$$

where y is the deflection from the neutral axis. For the cantilever beam for which $y(0) = y'(0) = 0$, express the equation as a pair of simultaneous first-order equations. Then with suitable numerical values of E, I, and M, determine the numerical solution for the deflection of the beam extending from $x = 0$ to $x = 5$ feet.

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3 Fredholm integral equations

3.1 Introduction

In this chapter, we shall be concerned with the Fredholm integral equations. We have already discussed in previous chapters the evolution of integral equations that have enormous applications in the physical problems. Many initial and boundary value problems can be transformed into integral equations. Problems in the mathematical physics are usually governed by the integral equations. There are quite a few categories of the integral equations. Singular integral equations are very useful in many physical problems such as elasticity, fluid mechanics, electromagnetic theory, and so on.

In 1825 Abel first encountered a singular integral equation in connection with the famous tautochrone problem, namely the determination of the shape of the curve with given end points along which a particle slides under the action of gravity in a given interval of time. An equation of an unknown function $u(x)$, of a single variable x in the domain $a \leq x \leq b$ is said to be an integral equation for $u(x)$, if $u(x)$ appears under the integral sign and the integral exists in some sense. For examples,

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \quad a \leq x \leq b \tag{3.1}$$

$$f(x) = \int_a^b K(x,t)u(t)dt, \quad a \leq x \leq b \tag{3.2}$$

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\{u(t)\}^2 dt, \quad a \leq x \leq b \tag{3.3}$$

are all Fredholm integral equations. Equation (3.1) is defined as the nonhomogeneous Fredholm linear integral equation of the second kind; equation (3.2) is the Fredholm linear integral equation of the first kind; and equation (3.3) is the Fredholm nonlinear integral equation of the second kind. In all these examples, $K(x,t)$ and $f(x)$ are known functions. $K(x,t)$ is called the kernel of the integral equation defined in the rectangle R, for which $a \leq x \leq b$ and $a \leq t \leq b$ and $f(x)$ is

called the forcing term defined in $a \leq x \leq b$. If $f(x) = 0$, then the equations are homogeneous. The functions u, f, K may be complex-valued functions. The linear and nonlinear integral equations are defined when the unknown function appears linearly or nonlinearly under the integral sign. The parameter λ is a known quantity.

3.2 Various types of Fredholm integral equations

Associated with the integral equation (3.1), the following meaning of various forms are used:

- If the kernel $K(x, t)$ is such that

$$\int_a^b \int_a^b |K(x,t)|^2 dxdt < \infty$$

  has a finite value, then we call the kernel a regular kernel and the corresponding integral is called a regular integral equation.
- If the kernel $K(x, t)$ is of the form

$$K(x,t) = \frac{H(x,t)}{|x-t|^\alpha}$$

  where $H(x, t)$ is bounded in R, $a \leq x \leq b$ and $a \leq t \leq b$ with $H(x, t) \neq 0$, and α is a constant such that $0 \leq \alpha \leq 1$, then the integral equation is said to be a weakly singular integral equation.
- If the kernel $K(x, t)$ is of the form

$$K(x,t) = \frac{H(x,t)}{x-t}$$

  where $H(x, t)$ is a differentiable function of (x, t) with $H(x, t) \neq 0$, then the integral equation is said to be a singular equation with Cauchy kernel where the integral $\int_a^b \frac{H(x,t)}{x-t} u(t) dt$ is understood in the sense of Cauchy Principal Value (CPV) and the notation $P.V. \int_a^b$ is usually used to denote this. Thus

$$P.V. \int_a^b \frac{u(t)}{x-t} dt = \lim_{\varepsilon \to 0} \left\{ \int_a^{x-\varepsilon} \frac{u(t)}{x-t} dt + \int_{x+\varepsilon}^b \frac{u(t)}{x-t} dt \right\}$$

- If the kernel $K(x, t)$ is of the form

$$K(x,t) = \frac{H(x,t)}{(x-t)^2}$$

where $H(x,t)$ is a differentiable function of (x,t) with $H(x,t) \neq 0$, then the integral equation is said to be a hyper-singular integral equation when the integral

$$\int_a^b \frac{H(x,t)}{(x-t)^2} u(t)dt$$

is to be understood as a hyper-singular integral and the notation $\oint_a^b$ is usually used to denote this. Thus,

$$\oint_a^b \frac{u(t)}{(x-t)^2} dt$$
$$\approx \lim_{\varepsilon \to 0} \left\{ \int_a^{x-\varepsilon} \frac{u(t)}{(x-t)^2} dt + \int_{x+\varepsilon}^b \frac{u(t)}{(x-t)^2} dt - \frac{u(x+\varepsilon)+u(x-\varepsilon)}{2\varepsilon} \right\}$$

- If the limits of the integral, a and b are constants, then it is a Fredholm integral equation. Otherwise, if a or b is a variable x, then the integral equation is called Volterra integral equation.

In the following section we shall discuss the various methods of solutions of the Fredholm integral equation.

3.3 The method of successive approximations: Neumann's series

The successive approximation method, which was successfully applied to Volterra integral equations of the second kind, can be applied even more easily to the basic Fredholm integral equations of the second kind:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{3.4}$$

We set $u_0(x) = f(x)$. Note that the zeroth approximation can be any selected real-valued function $u_0(x)$, $a \le x \le b$. Accordingly, the first approximation $u_1(x)$ of the solution of $u(x)$ is defined by

$$u_1(x) = f(x) + \lambda \int_a^b K(x,t)u_0(t)dt \tag{3.5}$$

The second approximation $u_2(x)$ of the solution $u(x)$ can be obtained by replacing $u_0(x)$ in equation (3.5) by the previously obtained $u_1(x)$; hence we find

$$u_2(x) = f(x) + \lambda \int_a^b K(x,t)u_1(t)dt. \tag{3.6}$$

This process can be continued in the same manner to obtain the *nth* approximation. In other words, the various approximations can be put in a recursive scheme given by

$$\begin{aligned} u_0(x) &= \text{any selective real valued function} \\ u_n(x) &= f(x) + \lambda \int_a^b K(x,t)u_{n-1}dt, \quad n \geq 1. \end{aligned} \tag{3.7}$$

Even though we can select any real-valued function for the zeroth approximation $u_0(x)$, the most commonly selected functions for $u_0(x)$ are $u_0(x)=0$, 1, or x. We have noticed that with the selection of $u_0(x)=0$, the first approximation $u_1(x)= f(x)$. The final solution $u(x)$ is obtained by

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{3.8}$$

so that the resulting solution $u(x)$ is independent of the choice of $u_0(x)$. This is known as Picard's method.

The Neumann series is obtained if we set $u_0(x)=f(x)$ such that

$$\begin{aligned} u_1(x) &= f(x) + \lambda \int_a^b K(x,t)u_0(t)dt \\ &= f(x) + \lambda \int_a^b K(x,t)f(t)dt \\ &= f(x) + \lambda\psi_1(x) \end{aligned} \tag{3.9}$$

where

$$\psi_1(x) = \int_a^b K(x,t)f(t)dt \tag{3.10}$$

The second approximation $u_2(x)$ can be obtained as

$$\begin{aligned} u_2(x) &= f(x) + \lambda \int_a^b K(x,t)u_1(t)dt \\ &= f(x)\lambda \int_a^b K(x,t)\,\{f(t) + \lambda\psi_1(t)\}\,dt \\ &= f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) \end{aligned} \tag{3.11}$$

where

$$\psi_2(x) = \int_a^b K(x,t)\psi_1(t)dt \tag{3.12}$$

Proceeding in this manner, the final solution $u(x)$ can be obtained

$$\begin{aligned} u(x) &= f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) + \cdots + \lambda^n\psi_n(x) + \cdots \\ &= f(x) + \sum_{n=1}^{\infty} \lambda^n\psi_n(x), \end{aligned} \tag{3.13}$$

where

$$\psi_n(x) = \int_a^b K(x,t)\psi_{n-1}dt \quad n \geq 1 \tag{3.14}$$

Series (3.13) is usually known as Neumann series. This infinite series can be shown to be absolutely and uniformly convergent but in this text we do not want to pursue this matter; rather the interested reader is referred to the standard textbook available in the literature.

Remark

It can be later seen that the Neumann series is identical with the Adomian decomposition method for the linear Fredholm integral equation. And the successive approximation is identical with the Picard's method. For ready reference we cite below:

Picard's method:

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{3.15}$$

Neumann's series method:

$$u(x) = f(x) + \lim_{n\to\infty} \sum_{k=1}^{n} \lambda^k\psi_k(x). \tag{3.16}$$

Example 3.1

Solve the Fredholm integral equation

$$u(x) = 1 + \int_0^1 xu(t)dt$$

by using the successive approximation method.

Solution

Let us consider the zeroth approximation is $u_0(x) = 1$, and then the first approximation can be computed as

$$
\begin{aligned}
u_1(x) &= 1 + \int_0^1 xu_0(t)dt \\
&= 1 + \int_0^1 xdt \\
&= 1 + x
\end{aligned}
$$

Proceeding in this manner, we find

$$
\begin{aligned}
u_2(x) &= 1 + \int_0^1 xu_1(t)dt \\
&= 1 + \int_0^1 x(1+t)dt \\
&= 1 + x\left(1 + \frac{1}{2}\right)
\end{aligned}
$$

Similarly, the third approximation is

$$
\begin{aligned}
u_3(x) &= 1 + x\int_0^1 \left(1 + \frac{3t}{2}\right) dt \\
&= 1 + x\left(1 + \frac{1}{2} + \frac{1}{4}\right)
\end{aligned}
$$

Thus, we get

$$
u_n(x) = 1 + x\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}\right\}
$$

and hence

$$
\begin{aligned}
u(x) &= \lim_{n\to\infty} u_n(x) \\
&= 1 + \lim_{n\to\infty} x\sum_{k=0}^{n} \frac{1}{2^k} \\
&= 1 + x\left(1 - \frac{1}{2}\right)^{-1} \\
&= 1 + 2x
\end{aligned}
$$

This is the desired solution.

Example 3.2

Use the successive approximation to solve the following Fredholm integral equation

$$u(x) = \sin x + \int_0^{\pi/2} \sin x \cos t u(t) dt.$$

Solution

Let us set $u_0(x) = 1$. Then the first approximation is

$$\begin{aligned} u_1(x) &= \sin x + \sin x \int_0^{\pi/2} \cos t \, dt \\ &= 2 \sin x \end{aligned}$$

The second approximation is

$$\begin{aligned} u_2(x) &= \sin x + \sin x \int_0^{\pi/2} 2 \sin t \cos t dt \\ &= 2 \sin x \end{aligned}$$

Proceeding in this way, it can be easily recognized that the sequence of solutions can be written without formal computation as

$$u_3(x) = 2 \sin x, u_4(x) = 2 \sin x \ldots, u_n(x) = 2 \sin x.$$

Hence the final solution is

$$u(x) = \lim_{n \to \infty} u_n(x) = 2 \sin x.$$

This is the desired solution. We shall see later that this solution can be obtained very easily by the direct computational method.

3.4 The method of successive substitutions

This method is almost analogous to the successive approximation method except that it concerns with the solution of the integral equation in a series form through evaluating single and multiple integrals. The solution by the numerical procedure in this method is huge compared to other techniques.

We assume that

$$u(x) = f(x) + \lambda \int_a^b K(x,t) u(t) dt \tag{3.17}$$

In equation (3.17), as usual $K(x,t) \neq 0$, is real and continuous in the rectangle R, for which $a \leq x \leq b$ and $a \leq t \leq b$; $f(x) \neq 0$ is real and continuous in the interval I, for which $a \leq x \leq b$; and λ, a constant parameter.

Substituting in the second member of equation (3.17), in place of $u(t)$, its value as given by the equation itself, yields

$$u(x) = f(x) + \lambda \int_a^b K(x,t)f(t)dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1)u(t_1)dt_1 dt \quad (3.18)$$

Here, again we substitute for $u(t_1)$ its value as given in equation (3.17). Thus, we get

$$\begin{aligned} u(x) &= f(x) + \lambda \int_a^b K(x,t)f(t)dt \\ &\quad + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1)f(t_1)dt_1 dt \\ &\quad + \lambda^3 \int_a^b K(x,t) \int_a^b K(t,t_1) \int_a^b K(t_1,t_2)u(t_2)dt_2 dt_1 dt \end{aligned} \quad (3.19)$$

Proceeding in this manner we obtain

$$\begin{aligned} u(x) &= f(x) + \lambda \int_a^b K(x,t)f(t)dt \\ &\quad + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1)f(t_1)dt_1 dt \\ &\quad + \lambda^3 \int_a^b K(x,t) \int_a^b K(t,t_1) \int_a^b K(t_1,t_2)f(t_2)dt_2 dt_1 dt \\ &\quad + \cdots \end{aligned} \quad (3.20)$$

We note that the series solution given in equation (3.20) converges uniformly in the interval $[a,b]$ if $\lambda M(b-a) < 1$ where $|K(x,t)| \leq M$. The proof is very simple and can be found in Lovitt [7]. From equation (3.20), it is absolutely clear that in this method unknown function $u(x)$ is replaced by the given function $f(x)$ that makes the evaluation of the multiple integrals simple and possible. The technique will be illustrated below with an example.

Example 3.3

Using the successive substitutions solve the Fredholm integral equation

$$u(x) = \cos x + \frac{1}{2} \int_0^{\pi/2} \sin x u(t)dt.$$

Solution

Here, $\lambda = \frac{1}{2}$, $f(x) = \cos x$, and $K(x,t) = \sin x$ and substituting these values in equation (3.20), yields

$$\begin{aligned}
u(x) &= \cos x + \frac{1}{2}\int_0^{\pi/2} \sin x \cos t dt + \frac{1}{4}\int_0^{\pi/2} \sin x \int_0^{\pi/2} \sin t \cos t_1 dt_1 dt \\
&\quad + \frac{1}{8}\int_0^{\pi/2} \sin x \int_0^{\pi/2} \sin t \int_0^{\pi/2} \sin t_1 \cos t_2 dt_2 dt_1 dt + \cdots \\
&= \cos x + \frac{1}{2}\sin x + \frac{1}{4}\sin x + \frac{1}{8}\sin x + \cdots \\
&= \cos x + \sin x
\end{aligned}$$

This result can be easily verified by the direct computational method to be considered in the later section.

3.5 The Adomian decomposition method

The decomposition method was recently introduced by Adomian [1] in a book written by him. The method has much similarity with the Neumann series as has been discussed in the previous section. The decomposition method has been proved to be reliable and efficient for a wide class of differential and integral equations of linear and nonlinear models. Like Neumann series method, the method provides the solution in a series form and the method can be applied to ordinary and partial differential equations and recently its use to the integral equations was found in the literature (see Ref. [9]). The concept of uniform convergence of the infinite series was addressed by Adomian ([2], [3]) and Adomian and Rach [4] for linear problems and extended to nonlinear problems by Cherruault *et al* [5] and Cherruault and Adomian [6]. In this book, we do not want to repeat the convergence problems.

In the decomposition method, we usually express the solution of the linear integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{3.21}$$

in a series form like regular perturbation series (see Van Dyke [8]) defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{3.22}$$

Substituting the decomposition equation (3.22) into both sides of equation (3.21) gives

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t) \left\{ \sum_{n=0}^{\infty} u_n(t) \right\} dt \tag{3.23}$$

The components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x) \ldots$ of the unknown function $u(x)$ are completely determined in a recurrence manner if we set

$$\begin{aligned}
u_0(x) &= f(x) \\
u_1(x) &= \lambda \int_a^b K(x,t) u_0(t) dt \\
u_2(x) &= \lambda \int_a^b K(x,t) u_1(t) dt \\
u_3(x) &= \lambda \int_a^b K(x,t) u_2(t) dt \\
\cdots\cdots &= \cdots\cdots \\
u_n(x) &= \lambda \int_a^b K(x,t) u_{n-1}(t) dt,
\end{aligned} \tag{3.24}$$

and so on. The main idea here like perturbation technique is to determine the zeroth decomposition $u_0(x)$ by the known function $f(x)$. Once $u_0(x)$ is known, then successively determine $u_1(x)$, $u_2(x)$, $u_3(x), \ldots$, and so on.

A compact recurrence scheme is then given by

$$u_0(x) = f(x) \tag{3.25}$$

$$u_{n+1}(x) = \lambda \int_a^b K(x,t) u_n(t) dt, \quad n \geq 1 \tag{3.26}$$

In view of equations (3.25) and (3.26), the components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x) \ldots$ follow immediately. Once these components are determined, the solution $u(x)$ can be obtained using the series (3.22). It may be noted that for some problems, the series gives the closed-form solution; however, for other problems, we have to determine a few terms in the series such as $u(x) = \sum_{n=0}^{k} u_n(x)$ by truncating the series at certain term. Because of the uniformly convergence property of the infinite series a few terms will attain the maximum accuracy.

Remark

Sometimes it may be useful to decompose the $f(x)$ term into $f(x)=f_1(x)+f_2(x)$ such that

$$\begin{aligned}
u_0(x) &= f_1(x)\\
u_1(x) &= f_2(x) + \lambda \int_a^b K(x,t)u_0(t)dt\\
u_2(x) &= \lambda \int_a^b K(x,t)u_1(t)dt,
\end{aligned}$$

and so on. This decomposition is called modified decomposition method. We illustrate them by examples.

Example 3.4

Solve the Fredholm integral equation

$$u(x) = e^x - 1 + \int_0^1 tu(t)dt$$

by the decomposition method.

Solution

The decomposition method is used here. We have

$$\begin{aligned}
u_0(x) &= e^x - 1\\
u_1(x) &= \int_0^1 tu_0(t)dt = \int_0^1 t(e^t - 1)dt = \frac{1}{2}\\
u_2(x) &= \int_0^1 tu_1(t)dt = \int_0^1 \frac{t}{2}dt = \frac{1}{4}\\
u_3(x) &= \int_0^1 tu_2(t)dt = \int_0^1 \frac{t}{4}dt = \frac{1}{8},
\end{aligned}$$

and so on. Hence the solution can be written at once

$$u(x) = e^x - 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = e^x.$$

By the modified decomposition method, we set that $u_0(x) = e^x$ and

$$u_1(x) = -1 + \int_0^1 tu_0(t)dt = -1 + \int_0^1 te^t dt = 0$$
$$u_2(x) = \int_0^1 tu_1(t)dt = 0$$
$$u_n(x) = 0$$

The solution is $u(x) = e^x$. By direct computation, we obtain $u(x) = e^x - 1 + \alpha$, where $\alpha = \int_0^1 tu(t)dt$. Using the value of $u(x)$ under the integral sign yields, $\alpha = \int_0^1 t(e^t - 1 + \alpha)dt = \frac{1}{2}(1 + \alpha)$. Hence the value of $\alpha = 1$. Thus, the solution is $u(x) = e^x$ which is identical with the decomposition method.

3.6 The direct computational method

There exists an efficient method to solve certain type of Fredholm integral equations and this method is usually known as the **direct computational method.** In this method, the kernel $K(x,t)$ must be separable in the product form such that it can be expressed as

$$K(x,t) = g(x)h(t) \tag{3.27}$$

Accordingly, the Fredholm equation can be expressed as

$$\begin{aligned} u(x) &= f(x) + \lambda \int_a^b K(x,t)u(t)dt \\ &= f(x) + \lambda g(x) \int_a^b h(t)u(t)dt \end{aligned} \tag{3.28}$$

The integral equation (3.28) is a definite integral and we set

$$\alpha = \int_a^b h(t)u(t)dt \tag{3.29}$$

where α is an unknown constant that needs to be determined. It follows that equation (3.28) can be written as

$$u(x) = f(x) + \lambda\alpha g(x) \tag{3.30}$$

It is thus obvious that the solution $u(x)$ is completely determined provided α is known from equation (3.29).

Remark

It is to be noted here that the direct computational method determines the exact solution in a closed form, rather than a series form provided the constant α has

been determined. The evaluation is completely dependent upon the structure of the kernel $K(x,t)$, and sometimes it may happen that the value of α may contain more than one. The computational difficulties may arise in determining the constant α if the resulting algebraic equation is of third order or higher. This kind of difficulty may arise in the nonlinear integral equation.

Example 3.5

Solve the linear Fredholm integral equation

$$u(x) = e^x - \frac{e}{2} + \frac{1}{2} + \frac{1}{2}\int_0^1 u(t)dt.$$

Solution

Let us set $\alpha = \int_0^1 u(t)dt$. Then we have

$$u(x) = e^x - \frac{e}{2} + \frac{1}{2} + \frac{\alpha}{2}$$

Replacing the value of $u(x)$ in the above integral yields

$$\begin{aligned}\alpha &= \int_0^1 \left(e^t - \frac{e}{2} + \frac{1}{2} + \frac{\alpha}{2}\right) dt \\ &= (e-1) + \left(\frac{1}{2} - \frac{e}{2}\right) + \frac{\alpha}{2}\end{aligned}$$

and this reduces to $\frac{\alpha}{2} = \frac{e}{2} - \frac{1}{2}$. Therefore, the solution is $u(x) = e^x$. This solution can be verified easily.

3.7 Homogeneous Fredholm equations

This section deals with the study of the homogeneous Fredholm integral equation with separable kernel given by

$$u(x) = \lambda \int_0^b K(x,t)u(t)dt \tag{3.31}$$

This equation is obtained from the second kind Fredholm equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt,$$

setting $f(x) = 0$.

It is easily seen that $u(x) = 0$ is a solution which is known as the trivial solution. We look forward to a nontrivial solution of equation (3.31). Generally speaking, the homogeneous Fredholm integral equation with separable kernel may have nontrivial solutions. We shall use the direct computational method to obtain the solution in this case.

Without loss of generality, we assume that

$$K(x,t) = g(x)h(t) \tag{3.32}$$

so that equation (3.31) becomes

$$u(x) = \lambda g(x) \int_a^b h(t)u(t)dt \tag{3.33}$$

Let us set

$$\alpha = \int_a^b h(t)u(t)dt \tag{3.34}$$

such that equation (3.33) becomes

$$u(x) = \lambda \alpha g(x) \tag{3.35}$$

We note that $\alpha = 0$ gives the trivial solution $u(x) = 0$, which is not our aim in this study. However, to determine the nontrivial solution of equation (3.31), we need to determine the value of the parameter λ by considering $\alpha \neq 0$. This can be achieved by substituting equation (3.35) into equation (3.34) to obtain

$$\alpha = \lambda \alpha \int_a^b h(t)g(t)dt \tag{3.36}$$

or equivalently,

$$1 = \lambda \int_a^b h(t)g(t)dt \tag{3.37}$$

which gives a numerical value for $\lambda \neq 0$ by evaluating the definite integral in equation (3.37). However, determining λ, the nontrivial solution given by equation (3.35) is obtained.

Remark

The particular nonzero values of λ that result from solving the algebraic system of equations are called the eigenvalues of the kernel. And corresponding to each eigenvalue we shall have solutions that are called **eigen solutions** or eigenfunctions. These solutions are nonzero solutions within a finite interval (a, b).

Example 3.6

Find the nontrivial solutions of the following Fredholm integral equation

$$u(x) = \lambda \int_0^{\pi/2} \cos x \sin t u(t) dt.$$

Solution

The equation is rewritten as

$$u(x) = \lambda \cos x \int_0^{\pi/2} \sin t u(t) dt$$

Let us set

$$\alpha = \int_0^{\pi/2} \sin t u(t) dt.$$

Then

$$u(x) = \lambda\alpha \cos x.$$

Hence the value of λ can be obtained putting the expression of $u(x)$ into the α integral which reduces to give for $\alpha \neq 0$, $\lambda = 1$. Thus, with this value of λ the solution is obtained as $u(x) = \alpha \cos x$ known as the eigen function where α is an arbitrary constant.

Example 3.7

Find the nontrivial solution of the following homogeneous Fredholm integral equation

$$u(x) = \lambda \int_0^{\pi/4} \sec^2 x u(t) dt.$$

Solution

If we set $\alpha = \int_0^{\pi/4} u(t)dt$ in the given equation, we have $u(x) = \lambda\alpha \sec^2 x$ and therefore, $\alpha = \lambda\alpha \int_0^{\pi/4} \sec^2 t dt = \lambda\alpha$. If $\alpha \neq 0$, then $\lambda = 1$, and hence, the solution of the homogeneous Fredholm equation is simply $u(x) = \alpha \sec^2 x$ and α is an arbitrary constant.

Example 3.8

Find the nontrivial solution of the Fredholm homogeneous integral equation

$$u(x) = \frac{2}{\pi}\lambda \int_0^{\pi} \cos(x-t) u(t) dt.$$

Solution

Given that

$$
\begin{aligned}
u(x) &= \frac{2}{\pi}\lambda \int_0^{\pi} \cos(x-t)u(t)dt \\
&= \frac{2}{\pi}\lambda \int_0^{\pi} \{\cos x \cos t + \sin x \sin t\}u(t)dt \\
&= \frac{2}{\pi}\lambda\alpha \cos x + \frac{2}{\pi}\lambda\beta \sin x
\end{aligned}
$$

where $\alpha = \int_0^{\pi} \cos tu(t)dt$ and $\beta = \int_0^{\pi} \sin tu(t)dt$. Hence using the value of $u(x)$ under the integral signs of α and β, we obtain a simultaneous algebraic equation given by

$$
\begin{aligned}
\alpha &= \frac{2\lambda}{\pi}\left\{\int_0^{\pi} \cos t(\alpha \cos t + \beta \sin t)dt\right\} \\
\beta &= \frac{2\lambda}{\pi}\left\{\int_0^{\pi} \sin t(\alpha \cos t + \beta \sin t)dt\right\}
\end{aligned}
$$

After performing the integrations and reduction, the values of α and β are found to be $\alpha = \lambda\alpha$ and $\beta = \lambda\beta$. If $\alpha \neq 0$ and $\beta \neq 0$, then $\lambda = 1$. Therefore, the desired solution is

$$u(x) = \frac{2}{\pi}(\alpha \cos x + \beta \sin x),$$

where α and β are arbitrary constants.

3.8 Exercises

Solve the following linear Fredholm integral equations:

1. $u(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xtu(t)dt.$
2. $u(x) = \sec^2 x + \lambda \int_0^1 u(t)dt.$
3. $u(x) = \sec^2 x \tan x - \lambda \int_0^1 u(t)dt.$
4. $u(x) = \cos x + \lambda \int_0^{\pi} xtu(t)dt.$
5. $u(x) = e^x + \lambda \int_0^1 2e^x e^t u(t)dt.$

Solve the following homogeneous integral equations:

6. $u(x) = \int_0^1 u(t)dt.$
7. $u(x) = \int_0^1 (-1)u(t)dt.$

8. $u(x) = \frac{1}{2}\int_0^{\pi} \sin xu(t)dt.$
9. $u(x) = \frac{3}{1000}\int_0^{10} xtu(t)dt.$
10. $u(x) = \frac{1}{e^x - 1}\int_0^1 2e^x e^t u(t)dt.$

Solve the following integral equations:

11. $u(x) = x + \lambda\int_0^1 (1 + x + t)u(t)dt.$
12. $u(x) = x + \lambda\int_0^1 (x - t)u(t)dt.$
13. $u(x) = x + \lambda\int_0^1 (x - t)^2 u(t)dt.$
14. $u(x) = x + \lambda\int_0^{\pi} (1 + \sin x \sin t)u(t)dt.$

Solve the following Fredholm equations by the decomposition method:

15. $u(x) = x + \sin x - x\int_0^{\pi/2} u(t)dt.$
16. $u(x) = 1 + \sec^2 x - \int_0^{\pi/4} u(t)dt.$
17. $u(x) = \frac{1}{1 + x^2} + 2x \sinh(\pi/4) - x\int_{-1}^{1} e^{\tan^{-1} t} u(t)dt.$

Find the nontrivial solutions for the following homogeneous Fredholm integral equations by using the *eigenvalues* and *eigenfunctions* concepts:

18. $u(x) = \lambda\int_0^1 xe^t u(t)dt.$
19. $u(x) = \lambda\int_0^1 2tu(t)dt.$
20. $u(x) = \lambda\int_0^{\pi/3} \sec x \tan tu(t)dt.$
21. $u(x) = \lambda\int_0^{\pi/4} \sec x \tan tu(t)dt.$
22. $u(x) = \lambda\int_0^1 \sin^{-1} xu(t)dt.$
23. $u(x) = \lambda\int_0^{\pi/2} \cos x \sin tu(t)dt.$
24. $u(x) = \frac{2}{\pi}\lambda\int_0^{\pi} \sin(x - t)u(t)dt.$

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4 Nonlinear integral equations

4.1 Introduction

In the previous chapters, we have devoted considerable time and effort in studying the solution techniques of different kinds of linear integral equations. In this chapter, we shall investigate the solution techniques for the nonlinear integral equations. We have already mentioned that nonlinear integral equations yield a considerable amount of difficulties. However, due to recent development of novel techniques it is now possible to find solutions of some types of nonlinear integral equations if not all. In general, the solution of the nonlinear integral equation is not unique. However, the existence of a unique solution of nonlinear integral equations with specific conditions is possible. As we know there is a close relationship between the differential equation and the integral equation. We shall see in the next section some classical development of these two systems and the methods of solutions.

We first define a nonlinear integral equation in general, and then cite some particular types of nonlinear integral equations. In general, a nonlinear integral equation is defined as given in the following equation:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)F(u(t))dt \tag{4.1}$$

and

$$u(x) = f(x) + \lambda \int_a^b K(x,t)F(u(t))dt \tag{4.2}$$

Equations (4.1) and (4.2) are called *nonlinear* Volterra integral equations and *nonlinear* Fredholm integral equations, respectively. The function $F(u(x))$ is nonlinear except $F =$ a constant or $F(u(x)) = u(x)$ in which case F is linear. If

$F(u(x)) = u^n(x)$, for $n \geq 2$, then the F function is nonlinear. To clarify this point we cite a few examples below:

$$u(x) = x + \lambda \int_0^x (x - t)u^2(t)dt \tag{4.3}$$

$$u(x) = x + \lambda \int_0^1 \cos xu^3(t)dt \tag{4.4}$$

Equations (4.3) and (4.4) are nonlinear Volterra and Fredholm integral equations, respectively.

4.2 The method of successive approximations

We have observed that solutions in finite closed form only apply to certain special types of differential equations. This is true for the integral equations also. If an equation does not belong to one of these categories, in which case analytical solutions are not possible to obtain, we need to use approximate methods. The phase plane method, with graphical depiction of Chapter 4 of Applied Numerical Analysis by Rahman [8], gives a good general idea of the nature of the solution, but it cannot be relied upon for numerical values.

In this chapter, we shall consider three approximate methods: the first one is the Picard's method to obtain successive algebraic approximations. (**E. T. Picard, Professor at the University of Paris, one of the most distinguished mathematicians of his time. He is well known for his researches on the Theory of Functions, and his Traite d' analysis is a standard textbook**). By putting numbers in these, we generally get excellent numerical results. Unfortunately, the method can only be applied to a limited class of equations, in which the successive integrations be easily performed.

The second method is the Adomian decomposition method. Adomian [1] recently developed the so-called Adomian decomposition or simply the decomposition method. The method was well introduced and well presented by Adomian in his recent books (Refs. [2], [3] and [4]). The method proved to be reliable and effective for a wider class of linear and nonlinear equations. This method provides the solution in a series form. The method was applied to ordinary and partial differential equations, and was rarely used for integral equations. The concept of convergence of the solution obtained by this method was addressed by Adomian in two books [1,3] and extensively by Cherruault *et al* (Ref. [5]) and Cherruault and Adomian [6] for nonlinear problems. With much confidence, however, the decomposition method can be successfully applied towards linear and nonlinear integral equations.

The third method, which is extreme numerical and of much more general applications, is due to Runge. (**C. Runge, Professor at the University of Gottingen, was an authority on graphical method.**) With proper precautions it gives good results in most cases, although occasionally it may involve a very large amount of arithmetical calculation. We shall treat several examples by these methods to enable their merits to be compared.

4.3 Picard's method of successive approximations

Consider the initial value problem given by the first-order nonlinear differential equation $\frac{du}{dx} = f(x, u(x))$ with the initial condition $u(a) = b$ at $x = a$. This initial value problem can be transformed to the nonlinear integral equation and is written as

$$u(x) = b + \int_a^x f(x, u(x))dx.$$

For a first approximation, we replace the $u(x)$ in $f(x, u(x))$ by b, for a second approximation, we replace it by the first approximation, for the third by the second, and so on. We demonstrate this method by examples.

Example 4.1

Consider the first-order nonlinear differential equation $\frac{du}{dx} = x + u^2$, where $u(0) = 0$ when $x = 0$. Determine the approximate analytical solution by Picard's method.

Solution

The given differential equation can be written in integral equation form as

$$u(x) = \int_0^x (x + u^2(x))dx.$$

Zeroth approximation is $u(x) = 0$.
First approximation: Put $u(x) = 0$ in $x + u^2(x)$, yielding

$$u(x) = \int_0^x xdx = \frac{1}{2}x^2.$$

Second approximation: Put $u(x) = \frac{x^2}{2}$ in $x + u^2$, yielding

$$u(x) = \int_0^x \left(x + \frac{x^2}{4}\right) dx = \frac{x^2}{2} + \frac{x^5}{20}.$$

Third approximation: Put $u = \frac{x^2}{2} + \frac{x^5}{20}$ in $x + u^2$, giving

$$\begin{aligned}
u(x) &= \int_0^x \left\{x + \left(\frac{x^2}{2} + \frac{x^5}{20}\right)^2\right\} dx \\
&= \int_0^x \left(x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}\right) dx \\
&= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}.
\end{aligned}$$

Proceeding in this manner, Fourth approximation can be written after a rigorous algebraic manipulation as

Fourth approximation:

$$u(x) = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{7x^{11}}{8800} + \frac{3x^{14}}{49280} + \frac{87x^{17}}{23936000} + \frac{x^{20}}{7040000} + \frac{x^{23}}{445280000},$$

and so on. This is the solution of the problem in series form, and it seems from its appearance the series is convergent.

Example 4.2

Find the solution of the coupled first-order nonlinear differential equations by converting them to nonlinear integral equations

$$\frac{du}{dx} = v; \frac{dv}{dx} = x^3(u+v)$$

subject to the initial conditions $u(0) = 1$, and $v(0) = \frac{1}{2}$ when $x = 0$.

Solution

The coupled differential equations can be written in the form of coupled integral equations as follows:

$$u(x) = 1 + \int_0^x v dx$$

$$v(x) = \frac{1}{2} + \int_0^x x^3(u+v)dx$$

Performing the integrations as demonstrated in the above example, we obtain

First approximation:

$$u(x) = 1 + \frac{x}{2}$$

$$v(x) = \frac{1}{2} + \frac{3x^4}{8}$$

Second approximation:

$$u(x) = 1 + \frac{x}{2} + \frac{3x^5}{40}$$

$$v(x) = \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}$$

Third approximation:

$$u(x) = 1 + \frac{x}{2} + \frac{3x^5}{40} + \frac{x^6}{60} + \frac{x^9}{192}$$
$$v(x) = \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64} + \frac{7x^9}{360} + \frac{x^{12}}{256},$$

and so on. Thus, the solution is given by the above expressions up to third order.

Example 4.3

Find the solution of the nonlinear second-order ordinary differential equation

$$u(x)u''(x) = (u'(x))^2$$

with the initial conditions $u(0) = 1$ and $u'(0) = 1$ at $x = 0$ by converting it to the integral equation.

Solution

The given equation can be transformed into a couple of first-order differential equations

$$u'(x) = v(x);\ v'(x) = \frac{v^2}{u}$$

with the initial conditions: $u(0) = 1$ and $v(0) = 1$ at $x = 0$. The integral equations are the following:

$$u(x) = 1 + \int_0^x v(x)dx$$
$$v(x) = 1 + \int_0^x \frac{v^2}{u}dx$$

By Picard's successive approximation method we obtain
First approximation:

$$u(x) = 1 + \int_0^x dx = 1 + x$$
$$v(x) = 1 + \int_0^x dx = 1 + x$$

Second approximation:

$$u(x) = 1 + \int_0^x (1 + x)dx = 1 + x + \frac{x^2}{2!}$$
$$v(x) = 1 + \int_0^x (1 + x)dx = 1 + x + \frac{x^2}{2!}$$

Third approximation:

$$u(x) = 1 + \int_0^x \left(1 + x + \frac{x^2}{2!}\right) dx = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$v(x) = 1 + \int_0^x \left(1 + x + \frac{x^2}{2!}\right) dx = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

So continuing in this way indefinitely we see that

$$\begin{aligned} u(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= e^x \end{aligned}$$

which is the desired solution of the given nonlinear initial value problem.

Remark

It is worth mentioning here that it can be easily seen that the general solution of this second-order nonlinear ODE has the solution $u(x) = e^{mx}$ where $-\infty < m < \infty$. For any real number of m, e^{mx} is a solution which means that it has infinitely many solutions of exponential type. Therefore, the solution is not unique. On the other hand, if we consider two particular solutions, namely e^{-x} and e^{2x} for $m = -1$ and $m = 2$, although they individually satisfy the differential equation, but the general solution formed by the linear combination $u(x) = c_1 e^{-x} + c_2 e^{2x}$ where c_1 and c_2 are arbitrary constants will never satisfy the differential equation. This phenomena is observed in the nonlinear problem. However, it is not the case when the problem is linear. This can be easily verified by considering the linear second-order ODE given by $u'' - u' - 2u = 0$ that has two particular solutions e^{-x} and e^{2x} as cited above and the linear combination is a solution also. (see Ref. [7]).

4.4 Existence theorem of Picard's method

In the previous section, we have studied the successive approximations of Picard's method. Mathematicians always hope that they would discover a method for expressing the solution of any differential equations in terms of a finite number of known functions, or their integrals. When it is realized that this is not possible, the question arises as to whether a differential equation in general has a solution at all, and, if it has, of what kind.

There are two distinct methods of discussing this question, but here we will discuss only one. The first method due to Picard has already been illustrated by examples in the previous section. We obtain successive approximations, which apparently tend to a limit. We shall now prove that these approximations really do tend to a limit and that this limit gives the solution. Thus, we prove the existence of a solution of a differential equation of a fairly general type. A theorem of this kind is called an *Existence theorem*. Picard's method is not difficult, so we will proceed

with it at once before saying anything about the second method. Our aim is now to prove that the assumptions made in obtaining these solutions were correct, and to state exactly the conditions that are sufficient to ensure correctness in equations.

Picard's method of successive approximations

If $\frac{du}{dx} = f(x,u)$ and $u = b$ and $x = a$, the successive approximations for the values of u as a function of x are

$$\begin{aligned} u_1 &= b + \int_a^x f(x,b)dx \\ u_2 &= b + \int_a^x f(x,u_1)dx \\ u_3 &= b + \int_a^x f(x,u_2)dx \\ \cdots\cdots &= \cdots\cdots \\ u_{n+1} &= b + \int_a^x f(x,u_n)dx, \end{aligned}$$

and so on. We have already explained the applications of this method in the examples of the previous section. We reproduce the solution of the example where $f(x,u) = x + u^2$: with $b = a = 0$, and find,

$$\begin{aligned} u_1 &= \frac{x^2}{2} \\ u_2 &= \frac{x^2}{2} + \frac{x^5}{20} \\ u_3 &= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400} \\ u_4 &= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{7x^{11}}{8800} + \frac{3x^{14}}{49280} \\ &\quad + \frac{87x^{17}}{23936000} + \frac{x^{20}}{7040000} + \frac{x^{23}}{445280000}. \end{aligned}$$

These functions appear to be tending to a limit, at any rate for sufficiently small values of x. It is the purpose of the present article to prove that this is the case, not merely in this particular example, but whenever $f(x,u)$ obeys certain conditions to be specified. These conditions are that, after suitable choice of the positive number h and k, we can assert that, for all values of x between $a-h$ and $a+h$, and for all values of u between $b-k$ and $b+k$, we can find positive numbers M and A so that (i) $|f(x,u)| < M$, and (ii) $|f(x,u) - f(x,u')| < A|u - u'|$, u and u' being any two values of u in the range considered. In our example $f(x,u) = x + u^2$, condition (i) is obviously satisfied, taking for M any positive number greater than $|a| + h + (|b| + k)^2$.

Also, $|(x+u^2)-(x+u'^2)| = |u^2-u'^2| = |u+u'||u-u'| \leq 2(|b|+k)|u-u'|$. So, condition (ii) is also satisfied, taking $A = 2(|b|+k)$.

Now returning to the general case, we consider the difference between the successive approximations. We know that $u_1 - b = \int_a^x f(x,b)dx$, by definition; but $|f(x,b)| < M$ by condition (i), so $|u_1 - b| \leq |\int_a^x f(x,b)dx| < |\int_a^x Mdx| \leq M|x-a| < Mh$.

Also in a similar manner, it is easy to show that

$$u_2 - u_1 = \int_a^x \{f(x,u_1) - f(x,b)\}\, dx$$

but we have

$$|f(x,u_1) - f(x,b)| < A|u_1 - b| < AM|x-a| \text{ by conditions (i) and (ii).}$$

And so we obtain

$$\begin{aligned} |u_2 - u_1| &< |\int_a^x AM(x-a)dx| \\ &< \frac{1}{2}AM(x-a)^2 \\ &< \frac{1}{2}AMh^2 \end{aligned}$$

Proceeding in this way, we can write

$$|u_n - u_{n-1}| < \frac{1}{n!}MA^{n-1}h^n.$$

Thus, the infinite series

$$\begin{aligned} b + Mh &+ \frac{1}{2}MAh^2 + \cdots + \frac{1}{n!}MA^{n-1}h^n + \cdots \\ &= b + \frac{M}{A}\left\{Ah + \frac{1}{2}(Ah)^2 + \cdots + \frac{1}{n!}(Ah)^n + \cdots\right\} \\ &= b + \frac{M}{A}[e^{Ah} - 1] \end{aligned}$$

is convergent for all values of h, A, and M.

Therefore, the infinite series

$$b + (u_1 - b) = (u_2 - u_1) + (u_3 - u_2) + \cdots + (u_n - u_{n-1}) + \cdots$$

each term of which is equal or less in absolute value than the corresponding term of the preceding, is still more convergent.

That is to say the sequence

$$
\begin{aligned}
u_1 &= b + (u_1 - b) \\
u_2 &= b + (u_1 - b) + (u_2 - u_1),
\end{aligned}
$$

and so on, tends to a definite limit, say $U(x)$ which is what we wanted to prove.

We must now prove that $U(x)$ satisfies the differential equation. At first sight, this seems obvious, but it is not so really, for we must not assume without proof that

$$\lim_{n\to\infty} \int_a^x f(x, u_{n-1})dx = \int_a^x f(x, \lim_{n\to\infty} u_{n-1})dx.$$

The student who understands the idea of uniform convergence will notice that the inequalities associated with this proof that we have used to prove the convergence of our series really proves its uniform convergence also. If, $f(x, u)$ is continuous, $u_1, u_2, \ldots$ etc., are continuous also, and U is uniformly convergent series of continuous functions; that is, U is itself continuous, and $U - u_{n-1}$ tends uniformly to zero as n increases.

Hence, condition (ii), $f(x, U) - f(x, u_{n-1})$ tends uniformly to zero. From this we deduce that

$$\int_a^x \{f(x, U) - f(x, u_{n-1})\}\, dx$$

tends to zero. Thus, the limits of the relation

$$u_n = b + \int_a^x f(x, u_{n-1})dx$$

is

$$U = b + \int_a^x f(x, U)dx;$$

therefore,

$$\frac{dU}{dx} = f(x, U),$$

and $U = b$ when $x = a$. This completes the proof.

4.5 The Adomian decomposition method

The decomposition method is similar to the Picard's successive approximation method. In the decomposition method, we usually express the solution $u(x)$ of the integral equation

$$u(x) = b + \int_a^x f(x, u)dx \tag{4.5}$$

in a series form defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{4.6}$$

Substituting the decomposition equation (4.6) into both sides of equation (4.5) yields

$$\sum_{n=0}^{\infty} u_n(x) = b + \int_a^x f\left(x, \sum_{n=0}^{\infty} u_n(x)\right) dx$$

The components $u_0(x), u_1(x), u_2(x), \ldots$ of the unknown function $u(x)$ are completely determined in a recurrence manner if we set

$$\begin{aligned}
u_0(x) &= b \\
u_1(x) &= \int_a^x f(x, u_0)dx \\
u_2(x) &= \int_a^x f(x, u_1)dx \\
u_3(x) &= \int_a^x f(x, u_2)dx,
\end{aligned}$$

and so on. The above decomposition scheme for determination of the components $u_0(x), u_1(x), u_2(x), \ldots$ of the solution $u(x)$ of equation (4.5) can be written in a recurrence form by

$$\begin{aligned}
u_0(x) &= b \\
u_{n+1}(x) &= \int_a^x f(x, u_n)dx
\end{aligned}$$

Once these components are known, the solution $u(x)$ of equation (4.1) is readily determined in series form using the decomposition equation (4.2). It is important to note that the series obtained for $u(x)$ frequently provides the exact solution in closed form. However, for certain problems, where equation (4.2) cannot be evaluated, a truncated series $\sum_{n=0}^{k} u_n(x)$ is usually used to approximate the solution $u(x)$ if numerical solution is desired. This is exactly what we have described in Picard's method. However, with this type of decomposition, we have found some drawbacks. Hence we propose in the following a simple decomposition method not exactly like the Picard method.

In this new decomposition process, we expand the solution function in a straightforward infinite series

$$\begin{aligned}
u(x) &= u_0(x) + u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots \\
&= \sum_{n=0}^{\infty} u_n(x)
\end{aligned} \tag{4.7}$$

assuming that the series converges to a finite limit as $n \to \infty$.

Next we expand the function $f(x,u)$ which contains the solution function $u(x)$ by Taylor's expansion about $u_0(x)$ keeping x as it is such that

$$\begin{aligned} f(x,u) &= f(x,u_0) + (u-u_0)f_u(x,u_0) + \frac{(u-u_0)^2}{2!}f_{uu}(x,u_0) \\ &= \frac{(u-u_0)^3}{3!}f_{uuu}(x,u_0) + \frac{(u-u_0)^4}{4!}f_{uuuu}(x,u_0) + \cdots \end{aligned} \tag{4.8}$$

We know that Taylor's expansion is absolutely and uniformly convergent in a given domain. Now using equation (4.7) into equation (4.8), yields

$$\begin{aligned} f(x,u) = f(x,u_0) &+ \sum_{n=1}^{\infty} u_n(x) f_u(x,u_0) \\ &+ \frac{1}{2!}\left\{\sum_{n=1}^{\infty} u_n(x)\right\}^2 f_{uu}(x,u_0) \\ &+ \frac{1}{3!}\left\{\sum_{n=1}^{\infty} u_n(x)\right\}^3 f_{uuu}(x,u_0) \\ &+ \frac{1}{4!}\left\{\sum_{n=1}^{\infty} u_n(x)\right\}^4 f_{uuuu}(x,u_0) + \cdots \end{aligned} \tag{4.9}$$

which can subsequently be written as

$$\begin{aligned} f(x,u) &= A_0(x) + A_1(x) + A_2(x) + A_3(x) + \cdots + A_n(x) + \cdots \\ &= \sum_{n=0}^{\infty} A_n(x) \end{aligned} \tag{4.10}$$

We define the different terms in $A_n(x,u)$ as follows:

$$\begin{aligned} A_0 &= f(x,u_0) \\ A_1 &= u_1 f_u(x,u_0) \\ A_2 &= u_2 f_u(x,u_0) + \tfrac{1}{2}u_1^2 f_{uu}(x,u_0) \\ A_3 &= u_3 f_u(x,u_0) + \tfrac{1}{2}(2u_1u_2) f_{uu}(x,u_0) + \tfrac{1}{6}u_1^3 f_{uuu}(x,u_0) \\ A_4 &= u_4 f_u(x,u_0) + \tfrac{1}{2}(2u_1u_3 + u_2^2) f_{uu}(x,u_0) \\ &\quad + \tfrac{1}{6}(3u_1^2u_2) f_{uuu}(x,u_0) + \tfrac{1}{24}u_1^4 f_{uuuu}(x,u_0). \end{aligned} \tag{4.11}$$

Substituting equation (4.10) and equation (4.6) into the integral equation

$$u(x) = b + \int_a^x f(x,u)\,dx \tag{4.12}$$

we obtain

$$\sum_{n=0}^{\infty} u_n(x) = b + \int_a^x \sum_{n=0}^{\infty} A_n(x)dx$$

or simply

$$u_0(x) + u_1(x) + u_2(x) + \cdots = b + \int_a^x [A_0(x) + A_1(x) + A_2(x) + \cdots]dx.$$

The components $u_0(x), u_1(x), u_2(x), \ldots$ are completely determined by using the recurrence scheme

$$\begin{aligned}
u_0(x) &= b \\
u_1(x) &= \int_a^x A_0(x)dx = \int_a^x A_0(t)dt \\
u_2(x) &= \int_a^x A_1(x)dx = \int_a^x A_1(t)dt \\
u_3(x) &= \int_a^x A_2(x)dx = \int_a^x A_2(t)dt \\
u_4(x) &= \int_a^x A_3(x)dx = \int_a^x A_3(t)dt \\
\cdots\cdots &= \cdots\cdots \\
u_{n+1}(x) &= \int_a^x A_n(x)dx = \int_a^x A_n(t)dt, n \geq 1.
\end{aligned} \tag{4.13}$$

Consequently, the solution of equation (4.11) in a series form is immediately determined by using equation (4.6). As indicated earlier the series may yield the exact solution in a closed form, or a truncated series $\sum_{n=1}^{k} u_n(x)$ may be used if a numerical approximation is desired. In the following example, we will illustrate the decomposition method as established earlier by Picard's successive approximation method.

Example 4.4

Solve the integral equation

$$u(x) = \int_0^x (x + u^2)dx = \frac{x^2}{2} + \int_0^x u^2(t)dt$$

by the decomposition method.

Solution

With the decomposition method, we can write the equation in series form

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = \frac{x^2}{2} + \int_0^x \sum_{n=0}^{\infty} A_n(t)dt$$

in which we can decompose our solution set as

$$\begin{aligned}
u_0(x) &= \frac{x^2}{2} \\
u_1(x) &= \int_0^x A_0(t)dt \\
u_2(x) &= \int_0^x A_1(t)dt \\
\cdots\cdots &= \cdots\cdots \\
u_n(x) &= \int_0^x A_{n-1}(t)dt
\end{aligned} \tag{4.14}$$

We know $f(u) = u^2$, and so $f'(u) = 2u$ and $f''(u) = 2$. All the higher-order derivatives will be zero. Thus, we obtain

$$\begin{aligned}
f(u_0) &= u_0^2 = \frac{x^4}{4} \\
f'(u_0) &= 2u_0 = x^2 \\
f''(u_0) &= 2
\end{aligned}$$

Thus, with these information, we obtain

$$\begin{aligned}
A_0(x) &= \frac{x^4}{4} \\
A_1(x) &= u_1 x^2 \\
A_2(x) &= u_2 x^2 + u_1^2 \\
A_3(x) &= u_3 x^2 + 2u_1 u_2.
\end{aligned}$$

Hence the different components of the series can be obtained as

$$u_0(x) = \frac{x^2}{2}$$

$$
\begin{aligned}
u_1(x) &= \int_0^x A_0(t)dt = \int_0^x \frac{t^4}{4}dt = \frac{x^5}{20} \\
u_2(x) &= \int_0^x A_1(t)dt = \int_0^x u_1(t)dt = \int_0^x u_1(t)t^2 dt \\
&= \int_0^x \left(\frac{t^5}{2}\right) t^2 dt = \frac{x^8}{160} \\
u_3(x) &= \int_0^x A_2(t)dt = \int_0^x \left\{u_2(t)t^2 + u_1^2(t)\right\} dt \\
&= \int_0^x \left\{\left(\frac{t^8}{160}\right) t^2 + \frac{t^{10}}{400}\right\} dt = \frac{7x^{11}}{8800}
\end{aligned}
$$

Thus, the solution up to the third order is given by

$$
\begin{aligned}
u(x) &= u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \\
&= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{7x^{11}}{8800} + \cdots .
\end{aligned} \tag{4.15}
$$

Example 4.5

Obtain a solution of the following initial value problem by the Picard method and verify your result by the method of decomposition:

$$
\frac{du}{dx} = x + u
$$

$$
\text{with} \quad u(0) = 0.
$$

Solution

(a) Picard's method:

Given that $\frac{du}{dx} = x + u$, and integrating with respect to x and using the initial condition we obtain

$$
u(x) = \int_0^x (x + u)dx.
$$

The *first approximation* is

$$
u_1(x) = \int_0^x xdx = \frac{x^2}{2}.
$$

The *second approximation* is

$$
u_2(x) = \int_0^x (x + u_1)dx = \int_0^x \left(x + \frac{x^2}{2}\right) dx = \frac{x^2}{2!} + \frac{x^3}{3!}.
$$

Proceeding in this manner, the *nth approximation* can be written at once

$$u_n(x) = \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!},$$

and so on. Thus, the desired solution is

$$\begin{aligned} u(x) &= \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots \\ &= e^x - (1+x). \end{aligned} \tag{4.16}$$

(b) The decomposition method

The given integral equation can be written as

$$u(x) = \int_0^x (x+u)dx = \frac{x^2}{2} + \int_0^x u(t)dt.$$

Let $u(x)$ be decomposed into a series as

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots = \sum_{n=0}^{\infty} u_n(x).$$

Substituting this series into the above integral equation,

$$\sum_{n=0}^{\infty} u_n(x) = \frac{x^2}{2} + \int_0^x \sum_{n=0}^{\infty} u_n(t)dt.$$

Now equating term by term, yields

$$\begin{aligned} u_0(x) &= \frac{x^2}{2} \\ u_1(x) &= \int_0^x u_0(t)dt = \int_0^x \frac{t^2}{2}dt = \frac{x^3}{3!} \\ u_2(x) &= \int_0^x u_1(t)dt = \int_0^x \frac{t^3}{3!}dt = \frac{x^4}{4!}, \\ \cdots\cdots &= \cdots\cdots \end{aligned}$$

and so on. Thus, the solution is

$$u(x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x - (1+x). \tag{4.17}$$

Hence we get the identical results from these two methods. The analytical solution of this initial value problem is $u(x) = e^x - (1+x)$.

Example 4.6

Find the solution of the initial value problem by (a) Picard's method, (b) the decomposition method, $\frac{du}{dx} = xu$, $u(0) = 1$.

Solution

(a) Picard's method

The initial value problem can be put in the integral equation form

$$u(x) = 1 + \int_0^x tu(t)dt.$$

The first approximation is

$$u_1(x) = 1 + \int_0^x t(1)dt = 1 + \frac{x^2}{2}.$$

The second approximation is

$$u_2(x) = 1 + \int_0^x t\left(1 + \frac{t^2}{2!}\right) dt = 1 + \frac{x^2}{2} + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2.$$

Proceeding in this manner, we can derive the *nth* approximation as

The nth approximation is

$$u_n(x) = 1 + \frac{x^2}{2} + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 + \cdots + \frac{1}{n!}\left(\frac{x^2}{2}\right)^n.$$

Thus, when $n \to \infty$

$$u(x) = \lim_{n\to\infty} u_n(x) = e^{x^2/2}. \tag{4.18}$$

(b) The decomposition method

We decompose the solution in a series as $u(x) = \sum_{n=0}^{\infty} u_n(x)$. Substituting this series into the integral equation, we obtain

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \left\{ t \sum_{n=0}^{\infty} u_n(t) \right\} dt.$$

Equating the different order of terms of this series, we construct a set of integral equations

$$\begin{aligned}
u_0(x) &= 1\\
u_1(x) &= \int_0^x tu_0(t)dt = \int_0^x t(1)dt = \frac{x^2}{2}\\
u_2(x) &= \int_0^x tu_1(t)dt = \int_0^x t\left(\frac{t^2}{2}\right)dt = \frac{x^4}{4.2} = \frac{1}{2!}\left(\frac{x^2}{2}\right)^2\\
\cdots\cdots &= \cdots\cdots\\
u_n(x) &= \frac{1}{n!}\left(\frac{x^2}{2}\right)^n.
\end{aligned}$$

Thus, the solution is

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = 1 + \frac{x^2}{2} + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 + \cdots = e^{x^2/2}. \tag{4.19}$$

Solutions, i.e. equations (4.18) and (4.19) are identical and agree with the analytical solution $u(x) = e^{x^2/2}$.

Remark

It is worth noting that the decomposition method encounters computational difficulties if the nonhomogeneous term $f(x)$ is not a polynomial in terms of x. When the $f(x)$ is not polynomial we use the modified decomposition method to minimize the calculations. In the following example, we shall illustrate the modified decomposition method for handling the Volterra type integral equation.

Example 4.7

Obtain the solution of the following nonlinear nonhomogeneous integral equation of Volterra type,

$$u(x) = x + \frac{x^5}{5} - \int_0^x tu^3(t)dt.$$

Solution

By inspection, it can be observed that $u(x) = x$ is a solution of this integral equation. Because $\int_0^x t(t^3)dt = \frac{x^5}{5}$, and hence $u(x) = x + \frac{x^5}{5} - \frac{x^5}{5} = x$.

By the modified decomposition method, this is easily accomplished. In this method, the solution is expressed in a series which is given by $u(x) = \sum_{n=0}^{\infty} u_n(x)$, and $f(u) = u^3$ such that $f'(u) = 3u^2, f''(u) = 6u$, and $f'''(u) = 6$, and all the higher-order derivatives greater than the third order are zero.

By Taylor's expansion about $u = u_0$, $f(u)$ can be expressed as

$$\begin{aligned} f(u) = f(u_0) &+ (u_1 + u_2 + u_3 + \cdots) f'(u_0) \\ &+ \frac{1}{2!}(u_1 + u_2 + u_3 + \cdots)^2 f''(u_0) \\ &+ \frac{1}{3!}(u_1 + u_2 + u_3 + \cdots)^3 f'''(u_0) \\ &+ \cdots\cdots \end{aligned}$$

where we define the decomposition coefficients as

$$\begin{aligned} A_0 &= f(u_0) = u_0^3 \\ A_1 &= u_1 f'(u_0) = 3u_0^2 u_1 \\ A_2 &= u_2 f'(u_0) + \frac{1}{2} u_1^2 f''(u_0) = u_2(3u_0^2) + u_1^2(3u_0) \\ A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3} u_1^3 f''(u_0) \\ &= u_3(3u_0^2) + u_1 u_2 (6u_0) + u_1^3 \\ \cdots &= \cdots \end{aligned}$$

Let us consider $u_0(x) = x$. Then the subsequent terms of the series can be obtained as follows:

$$\begin{aligned} u_0(x) &= x \\ u_1(x) &= \frac{x^5}{5} - \int_0^x tA_0(t)dt \\ &= \frac{x^5}{5} - \int_0^x t(t^3)dt = \frac{x^5}{5} - \frac{x^5}{5} = 0 \\ u_2(x) &= -\int_0^x tA_1(t)dt = 0 \qquad \text{because } u_1(x) = 0. \end{aligned}$$

Similarly, $u_n(x) = 0$, for $n \geq 2$. And hence the solution is $u(x) = x$.

But with the usual procedure, if we set the zeroth component $u_0(x) = x + \frac{x^5}{5}$, then the first component is

$$\begin{aligned} u_1(x) &= -\int_0^x tA_0(t)dt = -\int_0^x t\left(t + \frac{t^5}{5}\right)^3 dt \\ &= -\frac{x^5}{5} - \frac{x^9}{15} - \frac{3x^{13}}{325} - \frac{x^{17}}{2125}. \end{aligned}$$

The second component is given by

$$\begin{aligned} u_2(x) &= -\int_0^x tA_1(t)dt \\ &= \int_0^x 3t\left(t+\frac{t^5}{5}\right)^2\left(\frac{x^5}{5}+\frac{x^9}{15}+\frac{3x^{13}}{325}+\frac{x^{17}}{2125}\right)dt. \end{aligned}$$

It can be easily observed that cancelling the **noise** term $\frac{x^5}{5}$ and $-\frac{x^5}{5}$ between $u_0(x)$ and $u_1(x)$, and justifying that the remaining term of $u_0(x)$ satisfies the equation, leads to the exact solution $u(x)=x$.

Example 4.8

Use the decomposition method or the modified decomposition method to solve the following nonlinear Volterra integral equation by finding the exact solution or a few terms of the series solution, $u(x)= \sin x+\frac{1}{8}\sin(2x)-\frac{x}{4}+\frac{1}{2}\int_0^x u^2(t)dt$.

Solution

By inspection it can be easily seen that $u(x)= \sin x$ is an exact solution. Let us confirm the result by the modified decomposition method. For this reason we split $f(x)= \sin x+\frac{1}{8}\sin(2x)-\frac{x}{4}$ between the two components $u_0(x)$ and $u_1(x)$, and here we set $u_0(x)= \sin x$. Consequently, the first decomposition component is defined by

$$u_1(x)=\frac{1}{8}\sin(2x)-\frac{x}{4}+\frac{1}{2}\int_0^x A_0(t)dt.$$

Here, $A_0(x)=u_0^2= \sin^2 x$. Thus, we obtain

$$\begin{aligned} u_1(x) &= \frac{1}{8}\sin(2x)-\frac{x}{4}+\frac{1}{2}\int_0^x \sin^2 t dt \\ &= \frac{1}{8}\sin(2x)-\frac{x}{4}+\left(\frac{x}{4}-\frac{1}{8}\sin(2x)\right)=0 \end{aligned}$$

This defines the other components by $u_k(x)=0$, for $k\geq 1$. The exact solution $u(x)= \sin x$ follows immediately.

Example 4.9

Use the decomposition method or the modified decomposition method to solve the following nonlinear Volterra integral equation by finding the exact solution or a few terms of the series solution

$$u(x)=\tan x-\frac{1}{4}\sin 2x-\frac{x}{2}+\int_0^x \frac{dt}{1+u^2(t)}.$$

Solution

By inspection it can be easily seen that $u(x) = \tan x$ is an exact solution. Let us show this result by the modified decomposition method. To accomplish this we split $f(x) = \tan x - \frac{1}{4}\sin 2x - \frac{x}{2}$ between the two components $u_0(x)$ and $u_1(x)$, and we set $u_0(x) = \tan x$. Consequently, the first component is defined by

$$\begin{aligned} u_1(x) &= -\frac{1}{4}\sin 2x - \frac{x}{2} + \int_0^x A_0(t)dt \\ &= -\frac{1}{4}\sin 2x - \frac{x}{2} + \int_0^x \frac{1}{1+\tan^2 t}dt \\ &= -\frac{1}{4}\sin 2x - \frac{x}{2} + \int_0^x \cos^2 tdt \\ &= -\frac{1}{4}\sin 2x - \frac{x}{2} + \frac{1}{4}\sin 2x + \frac{x}{2} \\ &= 0. \end{aligned}$$

This defines that the other components by $u_k(x) = 0$, for $k \geq 1$. Hence the exact solution $u(x) = \tan x$ follows immediately.

In the following example we shall deal with Fredholm type nonlinear integral equation by using the direct computational method.

Example 4.10

Use the direct computational method to solve the given nonlinear Fredholm integral equation and verify with the method of decomposition

$$u(x) = \frac{7}{8}x + \frac{1}{2}\int_0^1 xtu^2(t)dt. \tag{4.20}$$

Solution

(a) Direct computational method

Setting

$$\alpha = \int_0^1 tu^2(t)dt \tag{4.21}$$

where α is a constant, the given integral equation can be written as

$$u(x) = \tfrac{7}{8}x + \tfrac{1}{2}x\alpha. \tag{4.22}$$

But $\alpha = \int_0^1 t\left(\frac{7}{8} + \frac{\alpha}{2}\right)^2 t^2 dt = \frac{1}{4}\left(\frac{7}{8} + \frac{\alpha}{2}\right)^2.$

Now solving this quadratic equation in α, we obtain $\alpha = \frac{1}{4}, \frac{49}{4}$. Accordingly, we get two real solutions of the given integral equation

$$u(x) = \frac{7x}{8} + \frac{x}{8} = x;$$

and

$$u(x) = \frac{7x}{8} + \frac{x}{2}\frac{49}{4} = 7x.$$

(b) The decomposition method

In this method the Adomian polynomials for the nonlinear term $f(u) = u^2$ are expressed as

$$\begin{aligned}
A_0(x) &= u_0^2 \\
A_1(x) &= 2u_0u_1 \\
A_2(x) &= 2u_0u_2 + u_1^2 \\
A_3(x) &= 2u_0u_3 + 2u_1u_2 \\
\cdots\cdots &= \cdots\cdots
\end{aligned}$$

where the different components are calculated from $f(u) = u^2$, $f'(u) = 2u$, $f''(u) = 2$, $f'''(u) = 0$. Under the recursive algorithm, we have

$$\begin{aligned}
u_0(x) &= \frac{7}{8}x \\
u_1(x) &= \frac{x}{2}\int_0^1 tA_0(t)dt = \frac{49}{512}x \\
u_2(x) &= \frac{x}{2}\int_0^1 tA_1(t)dt = \frac{343}{16384}x
\end{aligned}$$

and so on. The solution in the series form is given by

$$\begin{aligned}
u(x) &= \frac{7}{8}x + \frac{49}{512}x + \frac{343}{16384}x + \cdots \\
&\approx x.
\end{aligned}$$

This is an example where the exact solution is not obtainable; hence we use a few terms of the series to approximate the solution. We remark that the two solutions $u(x) = x$ and $u(x) = 7x$ were obtained in (a) by the direct computational method. Thus, the given integral equation does not have a unique solution.

Remark

It is useful to note that the direct computational method produces multiple solutions for nonlinear integral equation. By using Adomian decomposition method, multiple solutions of nonlinear integral equation, if exist, cannot be determined which

appears to be the drawback of this method. However, the decomposition method is easier to use.

Example 4.11

By using the decomposition method, find the solution of the initial value problem $u''(x)+2u'(x)+2\sin u=0$ subject to the initial conditions $u(0)=1, u'(0)=0$. This is a practical problem governed by the oscillation of a pendulum.

Solution

The given initial value problem can be put into Volterra integral equation

$$u(x) = 1 + 2x - 2\int_0^x u(t)dt - 2\int_0^x (x-t)\sin u(t)dt.$$

Let us consider the solution in a series form as $u(x) = \sum_{n=0}^{\infty} u_n(x)$, and $f(u) = \sin u$. Expanding $f(u) = \sin u$ by Taylor's expansion about $u = u_0$, yields

$$\begin{aligned} f(u) &= \sin u_0 + \sum_{n=1}^{\infty} u_n(x)\cos u_0 \\ &\quad - \frac{1}{2}\left(\sum_{n=1}^{\infty} u_n(x)\right)^2 \sin u_0 - \frac{1}{6}\left(\sum_{n=1}^{\infty} u_n(x)\right)^3 \cos u_0 + \cdots \\ &= A_0 + A_1 + A_2 + A_3 + \cdots \end{aligned}$$

where

$$\begin{aligned} A_0(x) &= \sin u_0 \\ A_1(x) &= u_1(x)\cos u_0 \\ A_2(x) &= u_2(x)\cos u_0 - \frac{1}{2}u_1^2(x)\sin u_0 \\ A_3(x) &= u_3(x)\cos u_0 - u_1(x)u_2(x)\sin u_0 - \frac{1}{6}u_1^3(x)\cos u_0 \\ \cdots\cdots &= \cdots\cdots \end{aligned}$$

Let us consider $u_0(x)=1$. Then the subsequent terms of the series can be obtained as follows:

$$\begin{aligned} u_0(x) &= 1 \\ u_1(x) &= 2x - 2\int_0^x u_0(t)dt - 2\int_0^x (x-t)A_0(t)dt \\ &= 2x - 2\int_0^x (1)dt - 2\int_0^x (x-t)\sin 1 dt \end{aligned}$$

$$
\begin{aligned}
&= 2x - 2x - x^2 \sin 1 \\
&= -x^2(\sin 1) \\
u_2(x) &= -2\int_0^x u_1(t)dt - 2\int_0^x (x-t)A_1(t)dt \\
&= 2\sin 1 \int_0^x t^2 dt + 2\sin 1 \cos 1 \int_0^x (x-t)t^2 dt \\
&= \frac{2}{3}(\sin 1)x^3 + (\sin 2)\int_0^x (x-t)t^2 dt \\
&= \frac{2}{3}(\sin 1)x^3 + \frac{1}{12}(\sin 2)x^4.
\end{aligned}
$$

Thus, up to the second-order decomposition, the solution is

$$
\begin{aligned}
u(x) &= u_0(x) + u_1(x) + u_2(x) \\
&= 1 - (\sin 1)x^2 + \tfrac{2}{3}(\sin 1)x^3 + \tfrac{1}{12}(\sin 2)x^4.
\end{aligned}
$$

Remark

By using the differential equation with its initial conditions, we can solve this problem using the decomposition method as follows: Equating the different order of decomposition yields

$$
\begin{aligned}
u_0'' + 2u_0' &= 0 \\
u_1'' + 2u_1' &= -2A_0 \\
u_2'' + 2u_2' &= -2A_1 \\
u_3'' + 2u_3' &= -2A_2 \\
\cdots\cdots &= \cdots\cdots \\
u_n'' + 2u_n' &= -2A_{n-1}
\end{aligned}
$$

with the initial conditions

$$
\begin{aligned}
u_0(0) &= 1, \quad u_0' = 0 \\
u_1(0) &= 0, \quad u_1' = 0 \\
u_2(0) &= 0, \quad u_2' = 0 \\
u_3(0) &= 0, \quad u_3' = 0 \\
\cdots\cdots &= \cdots\cdots \\
u_n(0) &= 0, \quad u_n' = 0
\end{aligned}
$$

The solutions of this set of initial value problems can be obtained as follows:

$$\begin{aligned}
u_0(x) &= 1 \\
u_1(x) &= \frac{\sin 1}{2}(1 - 2x - e^{-2x}) \\
u_2(x) &= \frac{\sin 2}{8}\left\{3 - 4x + 2x^2 - (3 + 2x)e^{-2x}\right\}
\end{aligned}$$

Thus, the solution up to second-order decomposition term is given by

$$\begin{aligned}
u(x) &= u_0(x) + u_1(x) + u_2(x) \\
&= 1 + \left(\frac{\sin 1}{2}\right)(1 - 2x - e^{-2x}) \\
&\quad + \frac{\sin 2}{8}\left\{3 - 4x + 2x^2 - (3 + 2x)e^{-2x}\right\}
\end{aligned}$$

It can be easily verified that the solutions obtained from the integral equation are identical with those obtained from the differential equation up to the order x^4.

Example 4.12

Find the solution of the Klein–Gordon partial differential equation of the following form with the given initial conditions

$$\begin{aligned}
u_{tt} - u_{xx} - 2u &= -2\sin x \sin t \\
u(x, 0) &= 0, \\
u_t(x, 0) &= \sin x
\end{aligned} \tag{4.23}$$

Solution

The term $2\sin x \sin t$ will be shown to be a **noise term**. To solve this problem we first define the linear operators

$$\begin{aligned}
L_t(u) &= \frac{\partial^2 u}{\partial t^2} \\
L_x(u) &= \frac{\partial^2 u}{\partial x^2}
\end{aligned} \tag{4.24}$$

Integrating the equation (4.23) with respect to t partially two times from 0 to t and using the initial conditions we obtain

$$u(x, t) = t\sin x + L_t^{-1}\left(L_x(u) + 2u\right) + L_t^{-1}(-2\sin x \sin t) \tag{4.25}$$

If $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$, then the various iterates can be determined as

$$\begin{aligned}
u_0(x,t) &= t\sin x + L_t^{-1}(-2\sin x\sin t) \\
&= t\sin x + \int_0^t\int_0^t (-2\sin x\sin t)dtdt \\
&= 2\sin x\sin t - t\sin x \\
&= \sin x\,(2\sin t - t)
\end{aligned} \tag{4.26}$$

The $u_1(x,t)$ term can be obtained from the following

$$\begin{aligned}
u_1(x,t) &= L_t\,\{L_x(u_0) + 2u_0\} \\
&= \int_0^t\int_0^t \{L_x(u_0) + 2u_0\}\,dtdt \\
&= \int_0^t\int_0^t \{2\sin x\sin t - t\sin x\}\,dtdt \\
&= -2\sin t\sin x + 2t\sin x - \frac{t^3}{3!}\sin x \\
&= \sin x\left(-2\sin t + 2t - \frac{t^3}{3!}\right)
\end{aligned} \tag{4.27}$$

In a like manner, we find

$$\begin{aligned}
u_2(x,t) &= L_t\,\{L_x(u_1) + 2u_1\} \\
&= \int_0^t\int_0^t \{L_x(u_1) + 2u_1\}\,dtdt \\
&= 2\sin x\sin t - 2t\sin x + \frac{t^3}{3!}\sin x - \frac{t^5}{5!}\sin x \\
&= \sin x\left(2\sin t - 2t + \frac{t^3}{3!} - \frac{t^5}{5!}\right)
\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}
u_2(x,t) &= L_t\,\{L_x(u_2) + 2u_2\} \\
&= \int_0^t\int_0^t \{L_x(u_2) + 2u_2\}\,dtdt \\
&= -2\sin x\sin t + 2t\sin x - \frac{t^3}{3!}\sin x + \frac{2t^5}{5!}\sin x - \frac{t^7}{7!}\sin x \\
&= \sin x\left(-2\sin t + 2t - \frac{t^3}{3!} + \frac{2t^5}{5!} - \frac{t^7}{7!}\right)
\end{aligned} \tag{4.29}$$

Upon summing the iterates, we observe that

$$\phi_4(x,t) = \sum_{i=0}^{3} u_i(x,t) = \sin x \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \right) \tag{4.30}$$

This explains the phenomena that $2\sin x \sin t$ is the self-cancelling **noise** term. Further cancelling the noise term we obtain inductively the exact solution to equation (4.23) given by

$$u(x,t) = \sin x \sin t. \tag{4.31}$$

Example 4.13

Consider the following nonlinear pendulum like ordinary differential equation

$$\frac{d^2u}{dt^2} = \sin u \tag{4.32}$$

with the following initial conditions:

$$u(0) = \pi, \quad \frac{du}{dt}(0) = -2. \tag{4.33}$$

Solution

We shall show, using decomposition method, how to obtain solutions that coincide with implicit solution of equations (4.32)–(4.33) given by

$$\sin \frac{u}{2} = \operatorname{sech} t.$$

Here, the nonlinear function is $N(u) = \sin u$. If we set $L_t(u) = \frac{d^2u}{dt^2}$, then equation (4.32) can be expressed in operator form after integrating two times with respect to time t from 0 to t

$$u(t) = \pi - 2t + L_t^{-1}N(u) = \pi - 2t + \int_0^t \int_0^t N(u(t))dtdt \tag{4.34}$$

Thus, writing

$$u(t) = \sum_{n=0}^{\infty} u_n(t)$$

$$N(u) = \sin u = \sum_{n=0}^{\infty} A_n(t),$$

the various iterates are given by

$$u_{n+1}(t) = \int_0^t \int_0^t A_n(t)dtdt, \quad n \geq 0 \tag{4.35}$$

with

$$u_0(t) = \pi - 2t \tag{4.36}$$

For $N(u) = \sin u$, we have the following formulas

$$\begin{aligned}
A_0 &= \sin u_0 \\
A_1 &= u_1 \cos u_0 \\
A_2 &= u_2 \cos u_0 - \frac{u_1^2}{2!} \sin u_0 \\
A_3 &= u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{u_1^3}{3!} \cos u_0 \\
\cdots\cdots &= \cdots\cdots
\end{aligned} \tag{4.37}$$

Therefore, since u_0 is known, equations (4.35)–(4.36) provide the series solution $\sum_{n=0}^{\infty} u_n$, where

$$\begin{aligned}
u_0 &= \pi - 2t \\
u_1 &= \int_0^t \int_0^t A_0 dtdt \\
u_2 &= \int_0^t \int_0^t A_1 dtdt \\
u_3 &= \int_0^t \int_0^t A_2 dtdt \\
\cdots\cdots &= \cdots\cdots \\
u_{n+1} &= \int_0^t \int_0^t A_n dtdt
\end{aligned} \tag{4.38}$$

Using equation (4.38) the various iterates are given as

$$\begin{aligned}
u_1 &= \tfrac{t}{2} - \tfrac{1}{4} \sin 2t \\
u_2 &= \tfrac{5}{32} t - \tfrac{1}{8} \sin 2t + \tfrac{1}{8} t \cos 2t - \tfrac{1}{64} \sin 2t \cos 2t,
\end{aligned} \tag{4.39}$$

and so on, where the identities $\cos(\pi - 2t) = -\cos 2t$ and $\sin(\pi - 2t) = \sin 2t$ are used. Higher iterates can be determined similarly. Upon combining the first six iterates and expanding in Taylor's series around $t = 0$, we obtain

$$u = \pi - 2t + \frac{2}{3!} t^3 - \frac{10}{5!} t^5 + \frac{61}{7!} t^7 - \frac{2770}{9!} t^9 + \frac{103058}{11!} t^{11} + \cdots \tag{4.40}$$

which coincides with Taylor expansion of $\sin\frac{u}{2} = \operatorname{sech} t$. We observe that each time an iterate is added, the Taylor expansion coincides to the next higher term.

Example 4.14

Solve the D'Alembert's wave equation with the given initial conditions by the decomposition method

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}$$

Solution

The D'Alembert's wave equation can be transformed into the integral equation by using the given initial conditions as follows:

$$\begin{aligned} u_t(x,t) &= \psi(x) + c^2 \int_0^t u_{xx} dt \\ u(x,t) &= \phi(x) + t\psi(x) + c^2 \int_0^t \int_0^t u_{xx} dtdt \end{aligned} \tag{4.41}$$

Consider the infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{4.42}$$

which is known as the decomposition series. Using this series solution into the equation (4.41) we have

$$\sum_{n=0}^{\infty} u_n(x,t) = \phi(x) + t\psi(x) + c^2 \int_0^t \int_0^t \sum_{n=0}^{\infty} (u_n)_{xx} dtdt \tag{4.43}$$

Now the various iterates are obtained as

$$\begin{aligned} u_0(x,t) &= \phi(x) + t\psi(x) \\ u_1(x,t) &= c^2 \int_0^t \int_0^t (u_0)_{xx} dtdt \\ u_2(x,t) &= c^2 \int_0^t \int_0^t (u_1)_{xx} dtdt \\ u_3(x,t) &= c^2 \int_0^t \int_0^t (u_2)_{xx} dtdt \end{aligned}$$

$$
\begin{aligned}
u_4(x,t) &= c^2 \int_0^t \int_0^t (u_3)_{xx} dt dt \\
\cdots\cdots &= \cdots\cdots \\
u_n(x,t) &= c^2 \int_0^t \int_0^t (u_{n-1})_{xx} dt dt \\
\cdots\cdots &= \cdots\cdots
\end{aligned}
$$

Performing the indicated integrations, we can write the solutions of each iterate as follows:

$$
\begin{aligned}
u_1(x,t) &= c^2 \left\{ \frac{t^2}{2!}\phi''(x) + \frac{t^3}{3!}\psi''(x) \right\} \\
u_2(x,t) &= (c^2)^2 \left\{ \frac{t^4}{4!}\phi^{(4)}(x) + \frac{t^5}{5!}\psi^{(4)}(x) \right\} \\
u_3(x,t) &= (c^2)^3 \left\{ \frac{t^6}{6!}\phi^{(6)}(x) + \frac{t^7}{7!}\psi^{(6)}(x) \right\} \\
\cdots\cdots &= \cdots\cdots \\
u_n(x,t) &= (c^2)^n \left\{ \frac{t^{2n}}{2n!}\phi^{(2n)}(x) + \frac{t^{2n+1}}{(2n+1)}\psi^{(2n)}(x) \right\} \\
\cdots\cdots &= \cdots\cdots
\end{aligned}
$$

Hence the solution can be written as

$$
\begin{aligned}
u(x,t) &= u_0 + u_1 + u_2 + \cdots + u_n = \cdots \\
&= \left\{ \sum_{n=0}^{\infty} \frac{(ct)^{2n}}{(2n)!}\phi^{(2n)}(x) \right\} + \left\{ \sum_{n=0}^{\infty} \frac{(ct)^{2n+1}}{c(2n+1)!}\psi^{(2n)}(x) \right\} \\
&= \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s) ds. \qquad (4.44)
\end{aligned}
$$

The compact form of the D'Alembert's wave solution, i.e. equation (4.44) can be very easily verified by using Taylor's expansion of $\phi(x \pm ct)$ and $\psi(x \pm ct)$ about ct with little manipulation.

Remark

The reader is referred to the work of Rahman ([7], [8]) for further information about D'Alembert's wave solution by the Characteristic method and also by the Fourier transform method.

4.6 Exercises

1. Use the decomposition method or otherwise to solve the following nonlinear Volterra integral equations:

$$(a)\quad u(x) = x^2 + \frac{1}{5}x^5 - \int_0^x u^2(t)dt.$$

$$(b)\quad u(x) = \tan x - \frac{1}{4}\sin(2x) - \frac{x}{2} + \int_0^x \frac{1}{1+u^2(t)}dt,\ x < \frac{\pi}{2}.$$

$$(c)\quad u(x) = e^x - \frac{1}{3}xe^{3x} + \frac{1}{3}x + \int_0^x xu^3(t)dt.$$

2. Use the direct computation method to solve the following nonlinear Fredholm integral equations:

$$(a)\quad u(x) = 1 + \lambda\int_0^1 tu^2(t)dt.$$

$$(b)\quad u(x) = 1 + \lambda\int_0^1 t^3u^2(t)dt.$$

$$(c)\quad u(x) = \sin x - \frac{\pi}{8} + \frac{1}{2}\int_0^{\frac{\pi}{2}} u^2(t)dt.$$

$$(d)\quad u(x) = x - \frac{5}{6} + \int_0^1 (u(t) + u^2(t))dt.$$

3. Use the decomposition method or otherwise to solve the following Fredholm nonlinear equations:

$$(a)\quad u(x) = 1 - \frac{x}{3} + \int_0^1 xt^2u^3(t)dt.$$

$$(b)\quad u(x) = 1 + \lambda\int_0^1 t^3u^2(t)dt,\ \lambda \le 1.$$

$$(c)\quad u(x) = \sinh x - 1 + \int_0^1 (\cosh^2(t) - u^2(t))dt.$$

$$(d)\quad u(x) = \sec x - x + \int_0^1 x(u^2(t) - \tan^2(t))dt.$$

4. The Klein–Gordon nonlinear hyperbolic partial differential equation with the initial conditions is given by

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^2 = 0;\quad a < x < b,\ t > t_0$$

$$u(x,0) = B\tan(kx)$$

$$u_t(x,0) = Bc\sec^2(kx),$$

where c, α, β, γ are constant, and $B = \sqrt{\frac{\beta}{\gamma}}$, $k = \sqrt{\frac{-\beta}{2(\alpha + c^2)}}$. Show by decomposition method that a closed-form solution is

$$u(x,t) = B\tan\left(k(x+ct)\right).$$

5. Consider the Klein–Gordon partial differential equation in the form

$$\begin{aligned} u_{tt} - u_{xx} + \frac{\pi^2}{4}u &= x^2\sin^2\left(\frac{\pi t}{2}\right) \\ u(x,0) &= 0 \\ u_t(x,0) &= \frac{\pi x}{2} \end{aligned}$$

Show by decomposition method that the exact solution is

$$u(x,t) = x\sin\left(\frac{\pi t}{2}\right).$$

6. Consider the hyperbolic equation

$$u_{tt} - \gamma^2 u_{xx} + c^2 u - \varepsilon^2 \sigma u^3 = 0,$$

where γ, c, σ are appropriate physical constants, with the initial conditions

$$u(x,0) = \cos kx, \quad u_t(x,0) = 0, \quad -\infty < x < \infty.$$

Show by decomposition method that the solution exists in the following manner for $(0 < \varepsilon \ll 1)$

$$\begin{aligned} u(x,t) &= \cos\omega t\cos kx + \varepsilon^2\left\{\frac{9\sigma}{32\omega}t\sin\omega t + \frac{3\sigma}{128\omega^2}(\cos\omega t - \cos 3\omega t)\right\}\cos kx \\ &\quad + \varepsilon^2\left\{\frac{3\sigma}{128\gamma^2k^2}(\cos\omega t - \cos\lambda t) + \frac{\sigma}{128c^2}(\cos\lambda t - \cos 3\omega t)\right\}\cos 3kx \\ &\quad + O(\varepsilon^3), \end{aligned}$$

where $\lambda^2 = 9\gamma^2k^2 + c^2$.

7. Using the solution of the D'Alembert's wave equation with the given initial conditions

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x), \end{aligned}$$

determine the wave solution if $\phi(x) = \frac{1}{1+x^2}$ and $\psi(x) = \sec^2 x$.

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5 The singular integral equation

5.1 Introduction

This chapter is concerned with the **singular integral equation** that has enormous applications in applied problems including fluid mechanics, bio-mechanics, and electromagnetic theory. An integral equation is called a singular integral equation if one or both limits of integration becomes infinite, or if the kernel $K(x,t)$, of the equation becomes infinite at one or more points in the interval of integration. To be specific, the integral equation of the first kind

$$f(x) = \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt, \tag{5.1}$$

or the integral equation of the second kind

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt, \tag{5.2}$$

is called singular if $\alpha(x)$, or $\beta(x)$, or both limits of integration are infinite. Moreover, the equation (5.1) or (5.2) is also called a singular equation if the kernel $K(x,t)$ becomes infinite at one or more points in the domain of integration. Examples of the first type of singular integral equations are given below.

$$u(x) = e^x + \int_0^{\infty} K(x,t)u(t)dt \tag{5.3}$$

$$\mathcal{F}\{u(x)\} = \int_{-\infty}^{\infty} e^{-i\omega x}u(x)dx \tag{5.4}$$

$$\mathcal{L}\{u(x)\} = \int_0^{\infty} e^{-sx}u(x)dx \tag{5.5}$$

The integral equations (5.4) and (5.5) are Fourier transform and Laplace transform of the function $u(x)$, respectively. In fact these two equations are Fredholm equations of the first kind with the kernel given by $K(x,\omega)=e^{-i\omega x}$ and $K(x,s)=e^{-sx}$, respectively. The reader is familiar with the use of Fourier and Laplace transforms in solving the ordinary and partial differential equations with constant coefficients. Equations (5.3)–(5.5) can be defined also as the improper integrals because of the limits of integration are infinite.

Examples of the second type of singular integral equations are given by the following:

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt \tag{5.6}$$

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt \tag{5.7}$$

$$u(x) = f(x) + \lambda \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt \tag{5.8}$$

where the singular behaviour in these examples is attributed to the kernel $K(x,t)$ becoming infinite as $x \to \infty$.

Remark

It is important to note that the integral equations (5.6) and (5.7) are called Abel's problems and generalized Abel's integral equations, respectively, after the name of the Norwegian mathematician Niels Abel who invented them in 1823 in his research of mathematical physics. Singular equation (5.8) is usually called the weakly-singular second kind Volterra integral equation.

In this chapter, we shall investigate the Abel's type singular integral equation, namely where the kernel $K(x,t)$ becomes infinite at one or more points of singularities in its domain of definition. We propose to investigate the following three types:

- Abel's problem,
- Generalized Abel's integral equations,
- The weakly-singular second kind Volterra-type integral equations.

5.2 Abel's problem

We have already established the integral equation of Abel's problem. The integral equation is given by equation (5.6), and we here reproducing it for clarity.

$$\int_0^x \frac{u(t)}{\sqrt{x-t}} dt = f(x)$$

The solution of this equation is attributed by using the Laplace transform method. Taking the Laplace transform of both sides of the above equation yields

$$\mathcal{L}\left\{\int_0^x \frac{u(t)}{\sqrt{x-t}}dt\right\} = \mathcal{L}\{f(x)\}.$$

Using the convolution theorem and after a little reduction, the transformed equation can be written in a simple form

$$\mathcal{L}\{u(x)\} = \frac{\sqrt{s}}{\sqrt{\pi}}\mathcal{L}\{f(x)\}.$$

Here, we have used the result of $\Gamma(\frac{1}{2})=\sqrt{\pi}$. The above transform cannot be inverted as it stands now. We rewrite the equation as follows:

$$\mathcal{L}\{u(x)\} = \frac{s}{\sqrt{\pi}}\left[\frac{1}{\sqrt{s}}\mathcal{L}\{f(x)\}\right].$$

Using the convolution theorem, it can be at once inverted to yield

$$\begin{aligned}
u(x) &= \frac{1}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{s\left[\frac{1}{\sqrt{s}}\mathcal{L}\{f(x)\}\right]\right\} \\
&= \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_0^x \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{x-t}}f(t)dt \\
&= \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}}dt
\end{aligned}$$

Note that the Leibnitz rule of differentiation cannot be used in the above integral. So, integrate the integral first and then take the derivative with respect to x. Then this gives

$$\begin{aligned}
u(x) &= \frac{1}{\pi}\frac{d}{dx}\left\{-2(\sqrt{x-t})f(t)|_0^x + 2\int_0^x \sqrt{x-t}f'(t)dt\right\} \\
&= \frac{1}{\pi}\left\{\frac{f(0)}{\sqrt{x}} + \int_0^x \frac{f'(t)}{\sqrt{x-t}}dt\right\}.
\end{aligned}$$

This is the desired solution of Abel's problem.

5.3 The generalized Abel's integral equation of the first kind

The integral equation is given by

$$\int_0^x \frac{u(t)dt}{(x-t)^\alpha} = f(x), \quad 0 < \alpha < 1 \tag{5.9}$$

Taking the Laplace transform of both sides with the help of convolution theorem, we obtain

$$\mathcal{L}\{x^{-\alpha}\}\mathcal{L}\{u(x)\} = \mathcal{L}\{f(x)\}$$

$$\text{or} \quad \frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\mathcal{L}\{u(x)\} = \mathcal{L}\{f(x)\} \tag{5.10}$$

Thus, rearranging the terms we have

$$\mathcal{L}\{u(x)\} = \frac{1}{\Gamma(1-\alpha)} s \left\{ \frac{1}{s^{\alpha}} \mathcal{L}\{f(x)\} \right\} \tag{5.11}$$

Using the convolution theorem of Laplace transform the equation (5.11) can be obtained as

$$\begin{aligned}
u(x) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx} \left\{ \int_0^x (x-t)^{\alpha} f(t)dt \right\} \\
&= \frac{\sin(\pi\alpha)}{\pi} \frac{d}{dx} \left\{ \frac{(x-t)^{\alpha}}{-\alpha} f(t)|_0^x + \frac{1}{\alpha} \int_0^x (x-t)^{\alpha} f'(t)dt \right\} \\
&= \frac{\sin(\pi\alpha)}{\pi} \frac{d}{dx} \left\{ \frac{(x)^{\alpha}}{\alpha} f(0) + \frac{1}{\alpha} \int_0^x (x-t)^{\alpha} f'(t)dt \right\} \\
&= \frac{\sin(\pi\alpha)}{\pi} \left\{ \frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(t)dt}{(x-t)^{1-\alpha}} \right\}
\end{aligned} \tag{5.12}$$

This is the desired solution of the integral equation. Here, it is to be noted that $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\sin(\pi\alpha)}{\pi}$. The definition of Gamma function is $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$.

5.4 Abel's problem of the second kind integral equation

The second kind Volterra equation in terms of Abel's integral equation is written as

$$\begin{aligned}
u(x) &= f(x) + \int_0^x K(x,t)u(t)dt \\
&= f(x) + \int_0^x \frac{u(t)dt}{\sqrt{x-t}}
\end{aligned} \tag{5.13}$$

The solution of this integral is attributed by the convolution theorem of Laplace transform. Taking the Laplace transform of both sides of the equation yields

$$\begin{aligned}
\mathcal{L}\{u(x)\} &= \mathcal{L}\{f(x)\} + \mathcal{L}\left\{\frac{1}{\sqrt{x}}\right\}\mathcal{L}\{u(x)\} \\
&= \mathcal{L}\{f(x)\} + \frac{\sqrt{\pi}}{\sqrt{s}}\mathcal{L}\{u(x)\}
\end{aligned}$$

and after reduction this can be expressed as

$$\begin{aligned}\mathcal{L}\{u(x)\} &= \left\{\frac{\sqrt{s}}{\sqrt{s}-\sqrt{\pi}}\right\}\mathcal{L}\{f(x)\} \\ &= \mathcal{L}\{f(x)\} + \left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\}\mathcal{L}\{f(x)\} \end{aligned} \tag{5.14}$$

The inversion of equation (5.14) is given by

$$u(x) = f(x) + \int_0^x g(t)f(x-t)dt \tag{5.15}$$

where $g(x) = \mathcal{L}^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\}$.

With reference to Rahman [1], the Laplace inverse of $\left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\}$ can be obtained from the formula

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s-a}-b}\right\} = e^{ax}\left\{\frac{1}{\sqrt{\pi x}} + be^{b^2x}erfc(-b\sqrt{x})\right\}.$$

In our problem, $a = 0$, $b = \sqrt{\pi}$ and so

$$\mathcal{L}^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\} = \sqrt{\pi}\left\{\frac{1}{\sqrt{\pi x}} + \sqrt{\pi}e^{\pi x}erfc(-\sqrt{\pi x})\right\}.$$

Here, it is noted that $erfc(-\sqrt{\pi x}) = erfc(\sqrt{\pi x})$. And hence

$$g(x) = \mathcal{L}^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\} = \sqrt{\pi}\left\{\frac{1}{\sqrt{\pi x}} + \sqrt{\pi}e^{\pi x}erfc(\sqrt{\pi x})\right\}.$$

Thus, the solution of the problem is given by equation (5.15).

5.5 The weakly-singular Volterra equation

The weakly-singular Volterra-type integral equations of the second kind, given by

$$u(x) = f(x) + \int_0^x \frac{\lambda}{\sqrt{x-t}}u(t)dt \tag{5.16}$$

appears frequently in many mathematical physics and chemistry applications such as heat conduction, crystal growth, and electrochemistry (see Riele [3]). It is to be noted that λ is a constant parameter. It is assumed that the function $f(x)$ is sufficiently smooth so that a unique solution to equation (5.16) is guaranteed. The kernel $K(x,t) = \frac{1}{\sqrt{x-t}}$ is a singular kernel.

We have already seen the use of convolution theorem of the Laplace transform method in the previous section. In this section, we shall use the decomposition method to evaluate this integral equation. To determine the solution we usually adopt the decomposition in the series form

$$u(x) = \sum_{0}^{\infty} u_n(x), \tag{5.17}$$

into both sides of equation (5.16) to obtain

$$\sum_{0}^{\infty} u_n(x) = f(x) \int_0^x \frac{\lambda}{\sqrt{x-t}} \left(\sum_{0}^{\infty} u_n(t) \right) dt \tag{5.18}$$

The components $u_0, u_1, u_2, \ldots$ are immediately determined upon applying the following recurrence relations

$$\begin{aligned}
u_0(x) &= f(x), \\
u_1(x) &= \int_0^x \frac{\lambda}{\sqrt{x-t}} u_0(t)dt, \\
u_2(x) &= \int_0^x \frac{\lambda}{\sqrt{x-t}} u_1(t)dt, \\
\cdots\cdots &= \cdots\cdots \\
u_n(x) &= \int_0^x \frac{\lambda}{\sqrt{x-t}} u_{n-1}(t)dt
\end{aligned} \tag{5.19}$$

Having determined the components $u_0(x), u_1(x), u_2(x), \ldots,$ the solution $u(x)$ of equation (5.16) will be easily obtained in the form of a rapid convergence power series by substituting the derived components in equation (5.17).

It is important to note that the phenomena of the self-cancelling noise terms, where like terms with opposite signs appears in specific problems, should be observed here between the components $u_0(x)$ and $u_1(x)$. The appearance of these terms usually speeds the convergence of the solution and normally minimizes the size of the computational work. It is sometimes convenient to use the modified decomposition method.

Example 5.1

Determine the solution of the weakly-singular Volterra integral equation of second kind

$$u(x) = \sqrt{x} + \frac{\pi x}{2} - \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt. \tag{5.20}$$

Solution

Using the recurrent algorithm, we set

$$u_0(x) = \sqrt{x} + \frac{\pi x}{2} \tag{5.21}$$

which gives

$$u_1(x) = -\int_0^x \frac{\sqrt{t} + \frac{\pi t}{2}}{\sqrt{x-t}}dt \tag{5.22}$$

The transformation $t = x\sin^2\theta$ carries equation (5.22) into

$$\begin{aligned} u_1(x) &= -\int_0^{\pi/2} (2x\sin^2\theta + \pi x^{3/2}\sin^3\theta)d\theta \\ &= -\frac{\pi x}{2} - \frac{2}{3}\pi x^{3/2}. \end{aligned} \tag{5.23}$$

Observing the appearance of the terms $\frac{\pi x}{2}$ and $-\frac{\pi x}{2}$ between the components $u_0(x)$ and $u_1(x)$, and verifying that the non-cancelling term in $u_0(x)$ justifies the equation (5.20) yields

$$u(x) = \sqrt{x} \tag{5.24}$$

the exact solution of the given integral equation.

This result can be verified by the Laplace transform method. By taking the Laplace transform of equation (5.20) yields

$$\begin{aligned} \mathcal{L}\{u(x)\} &= \mathcal{L}\{f(x)\} + \frac{\pi}{2}\mathcal{L}\{x\} - \mathcal{L}\left\{\frac{1}{\sqrt{x}}\right\}\mathcal{L}\{u(x)\} \\ &= \frac{\Gamma(3/2)}{s^{3/2}} + \frac{\pi}{2s^2} - \frac{\Gamma(1/2)}{\sqrt{s}}\mathcal{L}\{u(x)\} \end{aligned}$$

and after simplification we have

$$\begin{aligned} \mathcal{L}\{u(x)\} &= \frac{\sqrt{\pi}}{2s(\sqrt{s}+\sqrt{\pi})} + \frac{\pi}{2s^{3/2}(\sqrt{s}+\sqrt{\pi})} \\ &= \frac{\sqrt{\pi}}{2s^{3/2}}\left[\frac{\sqrt{s}+\sqrt{\pi}}{\sqrt{s}+\sqrt{\pi}}\right] \\ &= \frac{\sqrt{\pi}}{2s^{3/2}}. \end{aligned}$$

The inversion is simply

$$u(x) = \mathcal{L}^{-1}\left[\frac{\sqrt{\pi}}{2s^{3/2}}\right] = \sqrt{x}. \tag{5.25}$$

These two results are identical confirming the desired correct solution.

Example 5.2

Solve the Abel's problem

$$4\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt.$$

Solution

Taking the Laplace transform of both sides of the equation yields

$$4\mathcal{L}\{\sqrt{x}\} = \mathcal{L}\left\{\frac{1}{\sqrt{x}}\right\}\mathcal{L}\{u(x)\},$$

which reduces to

$$4\frac{\Gamma(3/2)}{s^{3/2}} = \frac{\Gamma(1/2)}{s^{1/2}}\mathcal{L}\{u(x)\}.$$

After simplifying it becomes

$$\mathcal{L}\{u(x)\} = 2/s,$$

the inversion of which yields $u(x) = 2$. This is the desired result.

5.6 Equations with Cauchy's principal value of an integral and Hilbert's transformation

We have seen in the previous section that a Volterra or Fredholm integral equation with a kernel of the type

$$K(x,t) = \frac{F(x,t)}{|x-t|^{\alpha}}, \quad (0 < \alpha < 1)$$

where F is bounded, can be transformed into a similar one with a bounded kernel. For this, the hypothesis that $\alpha < 1$ is essential. However, in the important case, $\alpha = 1$ in which the integral of the equation must be considered as a *Cauchy principal value integral*; the integral equation differs radically from the equations considered in the previous sections. It is important to define the *Cauchy principal value integral* at the beginning of this section.

The Cauchy principal value integral of a function $f(x)$ which becomes infinite at an interior point $x = x_0$ of the interval of integration (a, b) is the limit

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\left(\int_a^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^b\right) f(x)dx,$$

where $0 < \varepsilon \leq min(x_0 - a, b - x_0)$. If $f(x) = \frac{g(x)}{x - x_0}$, where $g(x)$ is any integrable function (in the sense of Lebesgue), then the above limit exists and is finite for

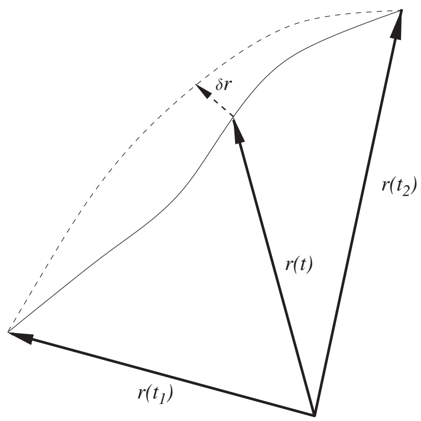

Figure 5.1: The actual path of a moving particle and possible variation of the path.

almost every x_0 in (a, b); and if $g(x)$ belongs to the class L_p with $p > 1$, the principal integral also belongs to L_p. (See, for example, Titchmarsh [4]).

One of the oldest results in this field consists of two reciprocity formulas which D. Hilbert deduced from the Poisson integral. He showed the following analysis to arrive at these formulas. We know that the Cauchy's integral formula in complex variables round a closed contour C is given by (see Rahman [1, 2])

$$\Phi(z) = \frac{1}{2\pi i} \oint_C \frac{\Phi(t)}{t - z} dt$$

Here, the function $\Phi(t)$ is analytic in the given domain and $t = z$ is a simple pole. Now, if we consider a semi-circular contour of infinite radius (see Figure 5.1) then we can evaluate the integral around this contour by using the residue calculus as follows:

$$\int_{C_R} \frac{\Phi(t)}{t - z} dt + \int_{-R}^{x-\varepsilon} \frac{\Phi(t)}{t - z} dt + \int_{C_\varepsilon} \frac{\Phi(t)}{t - z} dt + \int_{x+\varepsilon}^{R} \frac{\Phi(t)}{t - z} dt = 0. \tag{5.26}$$

If we let $R \to \infty$ and $\varepsilon \to 0$, then

$$\left| \int_{C_R} \frac{\Phi(t)}{t - z} dt \right| \to 0$$

and

$$\int_{C_\varepsilon} \frac{\Phi(t)}{t - z} dt = -\pi i \Phi(z),$$

and equation (5.26) reduces to the Cauchy's principal value (CPV)

$$\Phi(z) = \frac{1}{\pi i} P.V. \int_{-\infty}^{\infty} \frac{\Phi(t)}{t - z} dt \tag{5.27}$$

where $\Phi(z) = u(x,y) + iv(x,y)$. If the pole $z = x + i0$, then

$$\Phi(x + i0) = \frac{1}{\pi i} P.V. \int_{-\infty}^{\infty} \frac{u(t) + iv(t)}{t - x} dt$$

which implies that

$$u(x) + iv(x) = \frac{1}{\pi i} P.V. \int_{-\infty}^{\infty} \frac{u(t) + iv(t)}{t - x} dt$$

Now, equating the real and imaginary parts we obtain (we drop *P.V.* as we understand that the integrals are in the sense of Cauchy's principal value)

$$u(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(t)}{t - x} dt \tag{5.28}$$

$$v(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t)}{t - x} dt \tag{5.29}$$

These two formulas are defined as the Hilbert transform pairs, and usually denoted by

$$u(x) = \mathcal{H}\{v(t)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(t)}{t - x} dt \tag{5.30}$$

$$v(x) = -\mathcal{H}\{u(t)\} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t)}{t - x} dt \tag{5.31}$$

Remark

It is worth noting here that

$$u(x) = Re\{\Phi(x + i0)\} = \mathcal{H}\{Im(\Phi(x + i0))\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(t)}{t - x} dt$$

$$v(x) = Im\{\Phi(x + i0)\} = -\mathcal{H}\{Re(\Phi(x + i0))\} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t)}{t - x} dt$$

Next, we shall discuss four important theorems concerning the Hilbert transformations.

Theorem 5.1: (Reciprocity theorem)

If the function $\phi(x)$ belongs to the class $L_p (p > 1)$ in the basic interval $(-\infty, \infty)$, then formula (5.30) defines almost everywhere a function $f(x)$, which also belongs to L_p, whose Hilbert transform $\mathcal{H}[f]$ coincides almost everywhere with $-\phi(x)$.

That is, for any L_p function

$$\mathcal{H}(\mathcal{H}[\phi]) = -\phi. \tag{5.32}$$

Theorem 5.2: (Generalized Parseval's formula)

Let the functions $\phi_1(x)$ and $\phi_2(x)$ belong to the classes L_{p_1} and L_{p_2}, respectively. Then if

$$\frac{1}{p_1} + \frac{1}{p_2} = 1,$$

we have

$$\int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)dx = \int_{-\infty}^{\infty} \mathcal{H}[\phi_1(t)]\mathcal{H}[\phi_2(t)]dx.$$

Proof

The functions $\phi_1(x)$ and $\phi_2(x)$ belong to L_{p_1} and L_{p_2} class, respectively, which means that

$$\int_{-\infty}^{\infty} |\phi_1(x)|^{p_1} dx < K_1$$

and

$$\int_{-\infty}^{\infty} |\phi_2(x)|^{p_2} dx < K_2,$$

where K_1 and K_2 are finite real constants.

To prove this theorem, we consider

$$\begin{aligned}
\phi_1(x) &= v_1(x) \\
\phi_2(x) &= v_2(x) \\
\mathcal{H}\{\phi_1(t)\} &= u_1(x) \\
\mathcal{H}\{\phi_2(t)\} &= u_2(x)
\end{aligned}$$

Cauchy's integral formula gives us

$$\begin{aligned}
u_1(x) + iv_1(x) &= \frac{1}{\pi i} P.V. \int_{-\infty}^{\infty} \frac{u_1(t) + iv_1(t)}{t - x} dt \\
u_2(x) + iv_2(x) &= \frac{1}{\pi i} P.V. \int_{-\infty}^{\infty} \frac{u_2(t) + iv_2(t)}{t - x} dt
\end{aligned}$$

Equating the real and imaginary parts we obtain

$$\begin{aligned}
u_1(x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{v_1(t)}{t - x} dt = \mathcal{H}_x[v_1(t)] \\
u_2(x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{v_2(t)}{t - x} dt = \mathcal{H}_x[v_2(t)]
\end{aligned}$$

$$v_1(x) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{u_1(t)}{t-x} dt = -\mathcal{H}_x[u_1(t)]$$

$$v_2(x) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{u_2(t)}{t-x} dt = -\mathcal{H}_x[u_2(t)]$$

We know that

$$\phi_1(x) = v_1(x) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{u_1(t)}{t-x} dt$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)dx &= -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_2(x) \left[\frac{u_1(t)}{t-x} dt \right] dx \\
&= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} u_1(t) \int_{-\infty}^{\infty} \left[\frac{\phi_2(x)}{x-t} dx \right] dt \\
&= \int_{-\infty}^{\infty} u_1(t)u_2(t)dt \\
&= \int_{-\infty}^{\infty} \mathcal{H}_x[\phi_1(t)]\mathcal{H}_x[\phi_2(t)]dx
\end{aligned}$$

Hence Parseval's theorem

$$\int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)dx = \int_{-\infty}^{\infty} \mathcal{H}[\phi_1(t)]\mathcal{H}[\phi_2(t)]dx$$

is proved.

Theorem 5.3

Let $\Phi(x + iy)$ be an analytic function, regular for $y > 0$, which, for all values of y, satisfies the condition

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^p dx < K(p > 1),$$

where K is a positive constant. Then as $y \to +0$, $\Phi(x + iy)$ converges for almost all x to a limit function:

$$\Phi(x + i0) = u(x) + iv(x)$$

whose real and imaginary parts $u(x)$ and $v(x)$ are two L_p-functions connected by the reciprocity formulas in equations (5.28) and (5.29).

Hence, in particular, we have almost everywhere

$$Re\{|\Phi(\xi + i0)\} = \mathcal{H}_\xi[Im\{\Phi(\xi + i0)\}].$$

Conversely, given any real function $v(x)$ of the class L_p, if we put

$$u(x) = \mathcal{H}_x[v(t)],$$

then the analytic function $\Phi(z)$ corresponding to the pair (u, v) can be calculated by means of the formula

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(t) + iv(t)}{t - z} dt \quad (Im(z) > 0),$$

and it satisfies the condition given in Theorem 5.3.

To these theorems we add another, which plays an important role similar to that of the *convolution theorem* in the theory of Laplace transformations.

Theorem 5.4

Let the functions $\phi_1(x)$ and $\phi_2(x)$ belong to the classes L_{p_1} and L_{p_2}, respectively. Then if

$$\frac{1}{p_1} + \frac{1}{p_2} < 1,$$

i.e. if $p_1 + p_2 < p_1 p_2$, we have

$$\mathcal{H}\{\phi_1 \mathcal{H}[\phi_2] + \phi_2 \mathcal{H}[\phi_2]\} = \mathcal{H}[\phi_1]\mathcal{H}[\phi_2] - \phi_1\phi_2$$

almost everywhere.

Proof

To prove this theorem, we define the following analytic functions. Let

$$\Phi(x + i0) = u(x) + iv(x)$$

$$\Phi_1(x + i0) = u_1(x) + iv_1(x)$$

$$\Phi_2(x + i0) = u_2(x) + iv_2(x)$$

Define

$$\begin{aligned} \Psi(x + i0) &= \Phi_1(x + i0)\Phi_2(x + i0) \\ &= (u_1(x) + iv_1(x))(u_2(x) + iv_2(x)) \\ &= (u_1u_2 - v_1v_2) + i(u_1v_2 + u_2v_1) \end{aligned}$$

Hence we have

$$Re[\Psi(x + i0)] = u_1u_2 - v_1v_2$$

$$Im[\Psi(x + i0)] = u_1v_2 + u_2v_1$$

Therefore, by definition of Hilbert transform we have

$$\mathcal{H}[Im(\Psi(x+i0))] = Re[\Psi(x+i0)]$$

$$\text{or} \;\; \mathcal{H}[u_1v_2 + u_2v_1] = u_1u_2 - v_1v_2$$

Let us set

$$\begin{aligned} v_1 &= \phi_1(x) \\ v_2 &= \phi_2(x) \\ u_1 &= \mathcal{H}[\phi_1] \\ u_2 &= \mathcal{H}[\phi_2] \end{aligned}$$

Hence we obtain

$$\mathcal{H}[\phi_1\mathcal{H}\{\phi_2\} + \phi_2\mathcal{H}\{\phi_1\}] = \mathcal{H}\{\phi_1\}\mathcal{H}\{\phi_2\} - \phi_1\phi_2 \tag{5.33}$$

which is the required proof.

Note that by definition

$$\begin{aligned} u_1(x) = \mathcal{H}[v_1(t)] &= \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{v_1(t)}{t-x}dt \\ v_1(t) = \mathcal{H}^{-1}[u_1(x)] &= -\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{u_1(x)}{x-t}dx \\ u_2(x) = \mathcal{H}[v_2(t)] &= \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{v_2(t)}{t-x}dt \\ v_2(t) = \mathcal{H}^{-1}[u_2(x)] &= -\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{u_2(x)}{x-t}dx. \end{aligned}$$

Hence it is obvious that

$$\mathcal{H}\mathcal{H}[v(t) = \mathcal{H}[u(x)] = -v(t).$$

Theorem 5.5

The Hilbert transform of the derivative of a function is equivalent to the derivative of the Hilbert transform of the function, that is

$$\mathcal{H}\left[\frac{du}{dt}\right] = \frac{d}{dx}\mathcal{H}[u(t)].$$

Proof

By the definition of Hilbert transform we have

$$\mathcal{H}[u(t)] = \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{u(t)}{t-x}dt.$$

If we substitute $s = t - x$ such that $ds = dt$, then the right-hand side becomes

$$\mathcal{H}[u(t)] = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{u(s+x)}{s} ds,$$

and then apply the derivative to both sides

$$\begin{aligned}
\frac{d}{dx}\mathcal{H}[u(t)] &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{u'(s+x)}{s} ds \\
&= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{u'(t)}{t-x} dt \\
&= \mathcal{H}\left[\frac{du}{dt}\right].
\end{aligned}$$

This is the required proof.

Theorem 5.6

If we assume that $f(t)$ and $\mathcal{H}[f(t)]$ belong to L_1 class, then the Hilbert transform of $(tf(t))$ is given by

$$\mathcal{H}[tf(t)] = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)dt + x\mathcal{H}[f(t)].$$

Proof

Consider the Hilbert transform of $(tf(t))$

$$\begin{aligned}
\mathcal{H}[tf(t)] &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{tf(t)}{t-x} dt \\
&= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{(t-x+x)f(t)}{t-x} dt, \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)dt + x\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)dt + x\mathcal{H}[f(t)].
\end{aligned}$$

This is the required proof.

Example 5.3

Show that

(a) $\mathcal{H}[\sin t] = \cos x$,
(b) $\mathcal{H}[\cos t] = -\sin x$.

Solution

Both (a) and (b) can be proved if we consider the analytic function $\Phi(x+i0) = e^{ix} = \cos x + i\sin x$. $Re[\Phi(x+i0)] = \cos x$, and $Im[\Phi(x+i0) = \sin x$. Hence according to the definition of Hilbert transform, we have

$$\cos x + i\sin x = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\cos t + i\sin t}{t-x} dt$$

Now, equating the real and imaginary parts yields

$$\cos x = \mathcal{H}[\sin t] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin t}{t-x} dt$$

$$\sin x = -\mathcal{H}[\cos t] = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos t}{t-x} dt$$

Hence the proof follows.

The same problem can be directly solved by the semi-circular contour with the radius infinity. We consider the integral $\int_{-\infty}^{\infty} \frac{e^{it}}{t-x} dt$. Here, the integration is done around the closed semi-circular contour. By using the Cauchy's integral formula we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{it}}{t-x} dt &= \pi i(\text{residue at the pole } t = x) \\ &= \pi i e^{ix} \\ &= \pi i(\cos x + i\sin x)\end{aligned}$$

Equating the real and imaginary parts we obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos t}{t-x} dt = -\sin x$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin t}{t-x} dt = \cos x$$

Hence the results follow.

Example 5.4

Find the Hilbert transform of $\delta(t)$.

Solution

$$\begin{aligned}\mathcal{H}\{\delta(t)\} &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\delta(t)}{t-x} dt \\ &= \frac{1}{\pi}\left(-\frac{1}{x}\right) = -\frac{1}{\pi x}.\end{aligned}$$

Example 5.5

Find the Hilbert transform of the function $\frac{\sin t}{t}$.

Solution

We consider the function as $\frac{e^{ix}}{t}$ and integrate this function around the infinite semi-circular contour and use the residue calculus. The integral is $\int_{-\infty}^{\infty} \frac{e^{it}}{t(t-x)} dt$. Note that there are two poles, one at $t = 0$ and the other at $t = x$. Both the poles are on the real axis (the path of integration). So, we will get two residues R_0 and R_1 corresponding to the two poles, respectively.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{it}}{t(t-x)} dt &= \pi i(R_0 + R_1) \\ &= \pi i \left(\frac{-1}{x} + \frac{e^{ix}}{x} \right)\end{aligned}$$

Equating the real and imaginary parts we obtain

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\cos t/t)}{t-x} dt = -\frac{\sin x}{x}$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\sin t/t)}{t-x} dt = \frac{\cos x - 1}{x}$$

Thus,

$$\mathcal{H}\left\{ \frac{\cos t}{t} \right\} = -\frac{\sin x}{x},$$

and

$$\mathcal{H}\left\{ \frac{\sin t}{t} \right\} = \frac{\cos x - 1}{x}.$$

Example 5.6

Determine the Hilbert transform of $\frac{1}{1+t^2}$.

Solution

The integral to be considered is $\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)(t-x)}$. We consider the infinite semi-circular contour again. The poles are at $t = x$ on the real axis, i.e. on the path of integration, and the other pole is at $t = i$ inside the contour. The residue at $t = x$ is

$R_0 = \frac{1}{1+x^2}$; and the residue at $t = i$ is $R_1 = \frac{1}{2i(i-x)}$. Thus,

$$\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)(t-x)} = \pi i R_0 + 2\pi i R_1$$
$$= \pi i \left(\frac{1}{1+x^2}\right) + 2\pi i \left(\frac{1}{2i(i-x)}\right)$$

Equating the real part we obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)(t-x)} = -\frac{x}{1+x^2}.$$

Hence the

$$\mathcal{H}\left\{\frac{1}{1+t^2}\right\} = \frac{-x}{1+x^2}.$$

5.7 Use of Hilbert transforms in signal processing

Signal processing is a fast growing field in this cutting edge technology. The effectiveness in utilization of bandwidth and energy makes the process even faster. Signal processors are frequently used in equipment for radio, transportation, medicine, and production. Hilbert transform is a very useful technique in signal processing. In 1743, a famous Swiss mathematician named Leonard Euler (1707–1783) developed the formula $e^{ix} = \cos x + i \sin x$. One hundred and fifty years later the physicist Erthur E. Kennelly and the scientist Charles P. Steinmetz used this formula to introduce the complex notation of harmonic wave form in electrical engineering, that is $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$. In the beginning of the twentieth century, the German scientist David Hilbert (1862–1943) finally showed that the function $\sin(\omega t)$ is the Hilbert transform of $\cos(\omega t)$. This gives us the $\pm\pi/2$ phase-shift operator which is the basic property of the Hilbert transform.

A real function $f(t)$ and its Hilbert transform $\mathcal{H}\{f(\tau)\} = \hat{f}(t)$ are related to each other in such a way that they together create a so-called strong analytic signal. The strong analytic signal can be expressed with an amplitude and a phase where the derivative of the phase can be identified as the instantaneous frequency. The Fourier transform of the strong analytic signal gives us a one-sided spectrum in the frequency domain. It can be easily seen that a function and its Hilbert transform are orthogonal. This orthogonality is not always realized in applications because of truncations in numerical calculations. However, a function and its Hilbert transform have the same energy and the energy can be used to measure the calculation accuracy of the approximated Hilbert transform.

The Hilbert transform defined in the time domain is a convolution between the Hilbert transformer $\frac{1}{\pi t}$ and a function $f(t)$.

Definition 5.1

Mathematically, the Hilbert transform $\hat{f}(t)$ of a function $f(t)$ is defined for all t by

$$\hat{f}(t) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau,$$

where $P.V.$ stands for the Cauchy's principal value. It is normally not possible to evaluate the Hilbert transform as an ordinary improper integral because of the pole $\tau = t$. However, the $P.V.$ in front of the integral defines the Cauchy principal value and it expands the class of functions for which the Definition 5.1 exist.

The above definition of Hilbert transform can be obtained by using the Cauchy's integral formula using a semi-circular contour of infinite radius R and the real x-axis. If $f(z)$ is a function that is analytic in an open region that contains the upper-half plane and tends to zero at infinity at such a rate that contribution from the semi-circle vanishes as $R \to \infty$, then we have

$$P.V. \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi = \pi i f(x) \tag{5.34}$$

This result can be attributed to the residue calculus in which the residue at the pole $\xi = x$ is nothing but $f(x)$. If we express $f(x)$ as

$$f(x) = g(x) + ih(x),$$

on both sides of equation (5.34) with arguments on the real x-axis and equating real and imaginary parts, then we obtain for the real part

$$g(x) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{h(\xi)}{x-\xi} d\xi = -\mathcal{H}\{h(x)\},$$

and for the imaginary part

$$h(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{g(\xi)}{x-\xi} d\xi = \mathcal{H}\{g(x)\}. \tag{5.35}$$

From Definition 5.1, we have that $h(x)$ in equation (5.35) is the Hilbert transform of $g(x)$. We also note that $g(x) = \mathcal{H}^{-1}\{h(x)\}$ with $\mathcal{H}^{-1}$ as the inverse Hilbert transform operator. We see that $\mathcal{H}Ref(x) = Imf(x)$. Here, it is worth noting that the usual definition of Hilbert transform using the concept of the Cauchy's integral formula, i.e. equation (5.34) is $\mathcal{H}Imf(x) = Ref(x)$. It is hoped that the reader will not be confused with these two terminologies.

Definition 5.2

A complex signal $f(x)$ that fulfills the condition of the Cauchy's principal value is called a strong analytic signal. For a strong analytic signal $f(x)$ we have that $\mathcal{H}Ref(x) = Imf(x)$.

5.8 The Fourier transform

The Fourier transform is important in the theory of signal processing. When a function $f(t)$ is real, we only have to look on the positive frequency axis because it contains the complete information about the waveform in the time domain. Therefore, we do not need the negative frequency axis and the Hilbert transform can be used to remove it. This is explained below.

Let us define the Fourier transform $F(\omega)$ of a signal $f(t)$ by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{5.36}$$

This definition makes sense if $\int_{-\infty}^{\infty} |f(t)|dt$ exists. It is important to be able to recover the signal from its Fourier transform. To do that we define the inverse Fourier transform as

$$\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega. \tag{5.37}$$

If both f and F are integrable in the sense that $\int_{-\infty}^{\infty} |f(t)|dt < K_1$ and $\int_{-\infty}^{\infty} |F(\omega)|d\omega < K_2$ exist where K_1 and K_2 are two finite constants, then $f(t)$ is continuous and bounded for all real t and we have $\hat{f}(t) = f(t)$, that is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega. \tag{5.38}$$

Equation (5.38) is known as the inverse Fourier transform. The discrete form of the inversion formula is that if f belongs to $L^1(\Re)$, f is of bounded variation in the neighbourhood of t and f is continuous at t, then

$$f(t) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} F(\omega)e^{i\omega t}d\omega.$$

This means that equation (5.38) is to be interpreted as a type of Cauchy principal value. There is also a discrete formula for the Fourier transform when f belongs to $L^2(\Re)$, and in this case we define the Fourier transform as

$$F(\omega) = \lim_{N\to\infty}\int_{-N}^{N} f(t)e^{-i\omega t}dt.$$

Theorem 5.7

If f, g, and G belong to $L^1(\Re)$ or if f and g belong to $L^2(\Re)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega.$$

Proof

We know

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega$$

Then multiplying f by g^* and integrating both sides with respect to t from $-\infty$ to ∞, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} f(t) g^*(t) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \int_{-\infty}^{\infty} [g^*(t) e^{i\omega t} dt] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G^*(\omega) d\omega \end{aligned}$$

This is the required proof.

Note: If $f(t)$ is a real function then $f^*(t) = f(t)$ and hence

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(-\omega) e^{-i\omega t} d\omega \\ f^*(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(-\omega) e^{i\omega t} d\omega \end{aligned}$$

Therefore, it is obvious that $F(\omega) = F^*(-\omega)$ or $F(-\omega) = F^*(\omega)$ in the frequency domain and we see that F for negative frequencies can be expressed by F^* for positive one.

Theorem 5.8

If f(t) is a real function then

$$f(t) = \frac{1}{2\pi} \int_{0}^{\infty} [F^*(\omega) e^{-i\omega t} + F(\omega) e^{i\omega t}] d\omega.$$

Proof

The inverse Fourier transform of a real function $f(t)$ is given by

$$\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} F(\omega)e^{i\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{\infty} F(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\infty} F(-\omega)e^{-i\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{\infty} F(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\infty} [F^*(\omega)e^{-i\omega t} + F(\omega)e^{i\omega t}]d\omega
\end{aligned}$$

Hence the theorem is proved. This result implies that the positive frequency spectrum is sufficient to represent a real signal.

5.9 The Hilbert transform via Fourier transform

Let us define a function $S_f(\omega)$ which is zero for all negative frequency and $2F(\omega)$ for all positive frequencies

$$S_f(\omega) = F(\omega) + sgn(\omega)F(\omega) \tag{5.39}$$

where the function $sgn(\omega)$ is defined as

$$sgn(\omega) = \begin{cases} 1 & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ -1 & \text{for } \omega < 0 \end{cases}$$

and $F(\omega)$ is the Fourier transform of the real function $f(t)$. It can be easily visualized that the spectrum $S_f(\omega)$ is twice the measure of the spectrum $F(\omega)$, that is $S_f(\omega) = 2F(\omega)$. The inverse transform of $S_f(\omega)$ is therefore given by

$$s_f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)e^{i\omega t}d\omega = \frac{1}{\pi}\int_{0}^{\infty} F(\omega)e^{i\omega t}d\omega, \tag{5.40}$$

where $s_f(t)$ is a complex function of t in the form

$$s_f(t) = f(t) + ig(t) \tag{5.41}$$

From equations (5.39) and (5.41) we have that

$$\begin{aligned}
f(t) + ig(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [F(\omega) + sgn(\omega)\,F(\omega)]e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega + i\frac{1}{2\pi}\int_{-\infty}^{\infty} (-isgn(\omega))F(\omega)e^{i\omega t}d\omega
\end{aligned} \tag{5.42}$$

from which it is absolutely clear that as $\mathcal{F}\{f(t)\} = F(\omega)$, the Fourier transform of $g(t)$ must be $(-isgn(\omega))F(\omega)$. That means

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-isgn(\omega))F(\omega)e^{i\omega t}d\omega.$$

It is a standard result that the inverse Fourier transform of $(-isgn(\omega)) = \frac{1}{\pi t}$. Thus, using the convolution integral with this information we have

$$\begin{aligned}
g(t) &= \mathcal{F}^{-1}\left\{\mathcal{F}\left\{\frac{1}{\pi t}\right\}\mathcal{F}\{f(t)\}\right\} \\
&= f(t) * \frac{1}{\pi t} \\
&= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty}\frac{f(\tau)}{t-\tau}d\tau \\
&= \mathcal{H}\{f(t)\} = \hat{f}(t) \qquad (5.43)
\end{aligned}$$

and we see that $g(t)$ can be written as $\hat{f}(t)$ which is known as the Hilbert transform of $f(t)$. Further more $g(t)$ is real.

5.10 The Hilbert transform via the $\pm\pi/2$ phase shift

The Hilbert transform can be defined by the convolution of two functions $f(t)$ and $h(t)$ where f is a regular continuous function and h is the response of an impulse function, and usually represented by the formula

$$y(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau.$$

By the Fourier transform property, we see that

$$\mathcal{F}\{y(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{h(t)\}.$$

Thus, this impulsive response function $h(t)$ plays a very important role in producing the Hilbert transform. We shall illustrate below the ways to obtain this important function.

Let us consider a spectrum in the frequency domain defined by

$$H(\omega) = -isgn(\omega) = \begin{cases} -i = e^{-i\pi/2} & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ i = e^{i\pi/2} & \text{for } \omega < 0 \end{cases}$$

The $pm\pi/2$ phase shift is interpreted in the frequency domain as a multiplication with the imaginary value $\pm i$ as defined above. $h(\omega)$ is unfortunately not a property

of Fourier transform but the problem can be solved by expressing $H(\omega)$ as a limit of a bounded function $G(\omega)$, that is

$$G(\omega) = \begin{cases} -ie^{-a\omega} & \text{for } \omega > 0 \\ ie^{a\omega} & \text{for } \omega < 0 \end{cases}$$

where

$$\lim_{a \to 0} G(\omega) = H(\omega). \tag{5.44}$$

It is now possible to use the inverse Fourier transform on $G(\omega)$, thus

$$\begin{aligned} g(t) &= \mathcal{F}^{-1} G(\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{0} ie^{a\omega} e^{i\omega t} d\omega + \frac{1}{2\pi} \int_{0}^{\infty} -ie^{-a\omega} e^{i\omega t} d\omega \\ &= \frac{i}{2\pi} \int_{0}^{\infty} \left\{ e^{-(a+it)\omega} - e^{-(a-it)\omega} \right\} d\omega \\ &= \frac{i}{2\pi} \left(-\frac{e^{-(a+it)\omega}}{a+it} + \frac{e^{-(a-it)\omega}}{a-it} \right)_{0}^{\infty} \\ &= \frac{t}{\pi(a^2 + t^2)} \end{aligned}$$

where $g(t) \to h(t)$ when $a \to 0$ and the inverse Fourier transform of the impulse response of $H(\omega)$ is

$$h(t) = \lim_{a \to 0} g(t) = \lim_{a \to 0} \frac{t}{\pi(a^2 + t^2)} = \frac{1}{\pi t}.$$

A convolution between $f(t)$ and the impulse response $h(t)$ gives us

$$y(t) = \hat{f}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau,$$

where $\hat{f}(t)$ is known as the Hilbert transform. It is worth noting that this integral shall be considered as a principal value integral, a limit that corresponds to the limit in equation (5.44). To make a rigorous presentation of this problem we should apply the distribution theory but we shall not pursue this approach in this text. Thus, we can clearly define the Hilbert transform of $f(t)$ with the kernel $K(t, \tau) = \frac{1}{\pi(t-\tau)}$ as

$$\mathcal{H}\{f(\tau)\} = \hat{f}(t) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau,$$

where *P.V.* stands for the Cauchy's principal value as defined before.

5.11 Properties of the Hilbert transform

Some important properties of the Hilbert transform are discussed in this section. We assume that $F(\omega)$ does not contain any impulses at $\omega = 0$ and $f(t)$ is a real-valued function. Some of the properties are to be interpreted in a distributional sense.

5.11.1 Linearity

The Hilbert transform that is a Cauchy principal value function is expressed in the following form

$$\mathcal{H}\{f(t)\} = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau.$$

This definition is in accordance with the convolution integral of Fourier transform, and the notation for the Hilbert transform is used in a simple and understandable manner. Later we shall use another notation $\hat{f}(t) = \mathcal{H}\{f(t)\}$. We write the function $f(t) = c_1 f_1(t) + c_2 f_2(t)$, where c_1 and c_2 are two arbitrary constants. It is assumed that the Hilbert transform of $f_1(t)$ and $f_2(t)$ exists and therefore,

$$\begin{aligned}
\mathcal{H}\{f(t)\} &= \mathcal{H}\{c_1 f_1(t) + c_2 f_2(t)\} \\
&= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{c_1 f_1(\tau) + c_2 f_2(\tau)}{t-\tau} d\tau \\
&= c_1 \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f_1(\tau)}{t-\tau} d\tau + c_2 \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f_2(\tau)}{t-\tau} d\tau \\
&= c_1 \mathcal{H}\{f_1(t)\} + c_2 \mathcal{H}\{f_2(t)\}
\end{aligned}$$

This is the linearity property of the Hilbert transform.

5.11.2 Multiple Hilbert transforms and their inverses

In this section, we shall show that if we take the Hilbert transform twice on a real function it yields the same function it with a negative sign. Let us consider the Cauchy's integral formula

$$\begin{aligned}
f(t) + ig(t) &= \frac{1}{\pi i} P.V. \int_{-\infty}^{\infty} \frac{f(\tau) + ig(\tau)}{\tau - t} d\tau \quad \text{Cauchy's sense} \\
&= -\frac{1}{\pi i} P.V. \int_{-\infty}^{\infty} \frac{f(\tau) + ig(\tau)}{t - \tau} d\tau \quad \text{Fourier's sense} \\
&= \frac{i}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(\tau) + ig(\tau)}{t - \tau} d\tau \\
&= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{if(\tau) - g(\tau)}{t - \tau} d\tau
\end{aligned}$$

Equating the real and imaginary parts we obtain

$$f(t) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau} d\tau = -\mathcal{H}\{g(t)\}$$

$$g(t) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau = \mathcal{H}\{f(t)\}$$

These definition of Hilbert transform are due to Fourier.

Hence it is obvious that

$$-f(t) = \mathcal{H}\{g(t)\} = \mathcal{H}\mathcal{H}\{f(t)\}.$$

Thus, we see that the Hilbert transform used twice on a real function gives us the same real function but with altered sign.

$$\mathcal{H}\mathcal{H} = I,$$

with I as an identity operator. The Hilbert transform used four times on the same real function gives us the original function back

$$\mathcal{H}^2\mathcal{H}^2 = \mathcal{H}^4 = I \tag{5.45}$$

A more interesting property of multiple Hilbert transforms arises if we use the Hilbert transform three times, it yields

$$\mathcal{H}^3\mathcal{H} = I$$

which implies that

$$\mathcal{H}^3 = \mathcal{H}^{-1}.$$

This tells us that it is possible to use the multiple Hilbert transform to calculate the inverse Hilbert transform.

As we have seen before the Hilbert transform can be applied in the time domain by using the definition of Hilbert transform. In the frequency domain, we simply multiply the Hilbert transform operator $-isgn(\omega)$ to the function $F(\omega)$. By multiplying the Hilbert transform operator by itself we get an easy method to do multiple Hilbert transforms, that is

$$\mathcal{H}^n\{f(t)\} = (-isgn(\omega))^n F(\omega), \tag{5.46}$$

where n is the number of Hilbert transforms.

5.11.3 Derivatives of the Hilbert transform

Theorem 5.9

The Hilbert transform of the derivative of a function is equal to the derivative of the Hilbert transform, that

$$\mathcal{H}f'(t) = \frac{d}{dt}\hat{f}(t).$$

Proof

Consider the definition of the Hilbert transform

$$\begin{aligned}\hat{f}(t) &= \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{f(\tau)}{t-\tau}d\tau \\ &= \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{f(t-\tau)}{\tau}d\tau\end{aligned}$$

Now, differentiating with respect to t both sides we have

$$\begin{aligned}\frac{d}{dt}\hat{f}(t) &= \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{f'(t-\tau)}{\tau}d\tau \\ &= \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{f'(\tau)}{t-\tau}d\tau \\ &= \mathcal{H}f'(t).\end{aligned}$$

Hence the theorem is proved.

5.11.4 Orthogonality properties

Definition 5.3

A complex function is called Hermitian if its real part is even and its imaginary part is odd. From this definition, we infer that the Fourier transform $F(\omega)$ of a real function $f(t)$ is Hermitian.

Theorem 5.10

A real function $f(t)$ and its Hilbert transform $\hat{f}(t)$ are orthogonal if f, $\hat{f}$, and F belong to $L^1(\Re)$ or if f and $\hat{f}$ belong to $L^2(\Re)$.

Proof

By using the Parseval's identity we have that

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\hat{f}(t)dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)(-isgn(\omega)F(\omega))^{*}d\omega \\ &= \frac{i}{2\pi}\int_{-\infty}^{\infty} sgn(\omega)F(\omega)F^{*}(\omega)d\omega \\ &= \frac{i}{2\pi}\int_{-\infty}^{\infty} sgn(\omega)|F(\omega)|^{2}d\omega\end{aligned}$$

where $sgn(\omega)$ is an odd function and the fact that $F(\omega)$ is Hermitian gives us that $|F(\omega)|^2$ is an even function. We conclude that

$$\int_{-\infty}^{\infty} f(t)\hat{f}(t)dt = 0,$$

and therefore a real function and its Hilbert transform are orthogonal.

5.11.5 Energy aspects of the Hilbert transform

The energy of a function $f(t)$ is closely related to the energy of its Fourier transform $F(\omega)$. Theorem 5.7 which is known as the Parseval's theorem is called the Rayleigh theorem provided $g(t) = f(t)$. It will help us to define the energy of $f(t)$ and $F(\omega)$ as given below

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega. \tag{5.47}$$

Here, it is usual to assume that f belongs to$L^2(\Re)$ which means that E_f is finite. The same theorem can be used to define the energy of the Hilbert transform of $f(t)$ and $F(\omega)$, that is

$$E_{\hat{f}} = \int_{-\infty}^{\infty} \left|\hat{f}(t)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|-isgn(\omega)F(\omega)\right|^2 d\omega, \tag{5.48}$$

where $|-isgn(\omega)|^2 = 1$ except for $\omega = 0$. But, since $F(\omega)$ does not contain any impulses at the origin we get $E_{\hat{f}} = E_f$.

A consequence of equation (5.48) is that f in the space $L^2(\Re)$ indicates that $\hat{f}$ belongs to $L^2(\Re)$. The accuracy of the approximated Hilbert transform operator can be measured by comparing the energy in equation (5.47). However, a minor difference in energy always exists in real applications due to unavoidable truncation error.

5.12 Analytic signal in time domain

The Hilbert transform can be used to create an analytic signal from a real signal. Instead of studying the signal in the frequency domain it is possible to look at a rotating vector with an instantaneous phase $\theta(t)$ and an instantaneous amplitude $A(t)$ in the time domain, that is

$$z(t) = f(t) + i\hat{f}(t) = A(t)e^{i\theta(t)}.$$

This notation is usually called the polar notation where $A(t) = \sqrt{f^2(t) + \hat{f}^2(t)}$ and $\theta(t) = arctan\left\{\frac{\hat{f}(t)}{f(t)}\right\}$. If we express the phase by Taylor's expansion then

$$\theta(t) = \theta(t_0) + (t - t_0)\theta'(t_0) + R,$$

where R is small when t is close to t_0. The analytic signal becomes

$$z(t) = A(t)e^{i\theta(t)} = A(t)e^{i(\theta(t_0) - t_0\theta'(t_0))}e^{it\theta'(t_0)}e^{iR},$$

and we see that $\theta'(t_0)$ has the role of frequency if R is neglected. This makes it natural to introduce the notion of instantaneous angular frequency, that is

$$\omega(t) = \frac{d\theta(t)}{dt}.$$

The amplitude is simply

$$B(t) = A(t)e^{i(\theta(t_0) - t_0\theta'(t_0))}.$$

As for example if $z(t) = f(t) + i\hat{f}(t) = \cos\omega_0 t + i\sin\omega_0 t = A(t)e^{i\omega_0 t}$ such that $A(t) = \sqrt{\cos^2(\omega_0 t) + \sin^2(\omega_0 t)} = 1$ and the frequency is $\omega(t) = \omega_o$. In this particular case, we see that the instantaneous frequency is the same as the real frequency.

5.13 Hermitian polynomials

The numerical integration works fine on smooth functions that decrease rapidly at infinity. But it is efficient when a function decreases at a slow rate at infinity. In this section, we describe Hermite polynomials to calculate the Hilbert transform. First we need to take a look at the definition of the Hermite polynomials.

The successive differentiation of the Gaussian pulse e^{-t^2} generates the *nth* order Hermite polynomial which is defined by Rodrigues' formula as

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}.$$

It is also possible to calculate the Hermite polynomials by the recursion formula

$$H_n(t) = 2tH_{n-1}(t) - 2(n-1)H_{n-2}(t), \tag{5.49}$$

with $n = 1, 2, 3, \ldots$ and the start condition $H_0(t) = 1$.

Let us define the weighted Hermite polynomials that is weighted by the generating function e^{-t^2} such that

$$g_n(t) = H_n(t)e^{-t^2} = (-1)^n \frac{d^n}{dt^n} e^{-t^2}.$$

The weighted Hermite polynomials $g_n(t)$ do not represent an orthogonal set in L_2 since the scalar product

$$\int_{-\infty}^{\infty} g_n(t)g_m(t)dt = \int_{-\infty}^{\infty} H_n(t)H_m(t)e^{-2t^2}\,dt,$$

is in general different from zero when $n \neq m$. The solution is to replace the weighted function e^{-t^2} with $e^{-t^2/2}$, that is

$$\int_{-\infty}^{\infty} H_n(t)H_m(t)e^{-t^2}\,dt = \begin{cases} 0 & \text{for } n \neq m \\ 2^n n! \sqrt{\pi} & \text{for } n = m. \end{cases}$$

By that the weighted Hermitian polynomials fulfil the condition of orthogonality. The weighted polynomial $e^{-t^2/2}H_n(t)$ divided by their norm $\sqrt{2^n n! \sqrt{\pi}}$ yields a set of orthonormal functions in L_2 and is called the Hermite function

$$\varphi_n(t) = \frac{e^{-t^2/2}H_n(t)}{\sqrt{2^n n! \sqrt{\pi}}}. \tag{5.50}$$

If we combine equations (5.48) and (5.49), we get the recurrence algorithm

$$\varphi_n(t) = \sqrt{\frac{2(n-1)!}{n!}}\, t\varphi_{n-1}(t) - (n-1)\sqrt{\frac{(n-2)!}{n!}}\, \varphi_{n-2}(t), \tag{5.51}$$

which can be used to derive the Hilbert transform of the Hermite functions by applying the **multiplication by** t theorem.

Theorem 5.11

If we assume that $f(t)$ and $\hat{f}(t)$ belong to L_1 then the Hilbert transform of $tf(t)$ is given by the equation

$$\mathcal{H}\{tf(t)\} = t\hat{f}(t) - \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)d\tau.$$

The second term on the right-hand side with integral is a constant defined by the function $f(t)$. For odd functions this constant equal to zero.

Proof

Consider the Hilbert transform of $tf(t)$

$$\begin{aligned}
\mathcal{H}\{tf(t)\} &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\tau f(\tau)}{t-\tau} d\tau \\
&= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{(t-\tau) f(t-\tau)}{\tau} d\tau \\
&= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{t f(t-\tau)}{\tau} d\tau - \frac{1}{\pi} \int_{-\infty}^{\infty} f(t-\tau) d\tau \\
&= t\mathcal{H}\{f(t)\} - \frac{1}{\pi} \int_{-\infty}^{\infty} f(t-\tau) d\tau,
\end{aligned}$$

and the theorem is proved. From Theorem 5.11 and equation (5.51) we have that

$$\begin{aligned}
\mathcal{H}\{\varphi_n(t)\} &= \hat{\varphi}(t) \\
&= \sqrt{\frac{2(n-1)!}{n!}} \left\{ t\hat{\varphi}(t) - \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi_{n-1}(\eta) d\eta \right\} \\
&\quad - (n-1)\sqrt{\frac{(n-2)!}{n!}} \hat{\varphi}_{n-2}(t), \qquad (5.52)
\end{aligned}$$

where $n = 1, 2, 3, \ldots$. The first term $\varphi_0(t)$ can be obtained by using the Fourier transform on the equation

$$\varphi_0(t) = \pi^{-\frac{1}{4}} e^{-t^2/2}.$$

Thus, the Fourier transform of $\varphi_0(t) = \sqrt{2}\pi^{\frac{1}{4}} e^{-\omega^2/2}$.

In the frequency domain, we multiply the Hermite function $\varphi_0(t)$ by the Hilbert transform $-isgn(\omega)$ and finally we use the inverse Fourier transform to get the Hilbert transform of the Hermite function, that is

$$\mathcal{H}\{\varphi_0(t)\} = \hat{\varphi}_0(t) = \sqrt{2}\pi^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-\omega^2/2}(-isgn(\omega))e^{i\omega t} d\omega.$$

Since $sgn(\omega)e^{-\frac{\omega^2}{2}}$ is odd we have

$$\hat{\varphi}_0(t) = 2\sqrt{2}\pi^{\frac{1}{4}} \int_{0}^{\infty} e^{-\frac{\omega^2}{2}} \sin(\omega t) d\omega, \qquad (5.53)$$

which can be used in equation (5.52) to derive the rest of the Hilbert transforms.

Example 5.7

Determine the Hilbert transform of the Gaussian pulse e^{-t^2} by using Hermite polynomial.

Solution

To determine the Hilbert transform of the Gaussian pulse we need to first get the Fourier transform and then multiply this transform by $(-isgn(\omega))$ and obtain the inverse Fourier transform which will result in the Hilbert transform. Let us do it.

$$
\begin{aligned}
\mathcal{F}\{e^{-t^2}\} &= \int_{-\infty}^{\infty} e^{-t^2} e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-(t^2+i\omega t)} dt \\
&= e^{-\frac{\omega^2}{4}} \int_{-\infty}^{\infty} e^{-\left(t+\frac{i\omega}{2}\right)^2} dt \\
&= e^{-\frac{\omega^2}{4}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta \\
&= \sqrt{\pi} e^{-\frac{\omega^2}{4}}
\end{aligned}
$$

Now, we determine the Hilbert transform

$$
\begin{aligned}
\mathcal{H}\{e^{-t^2}\} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (-isgn(\omega))(\sqrt{\pi} e^{-\frac{\omega^2}{4}}) e^{i\omega t} d\omega \\
&= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} sgn(\omega) e^{-\frac{\omega^2}{4}} [\sin(\omega t) - i\cos(\omega t)] d\omega \\
&= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\frac{\omega^2}{4}} \sin(\omega t) d\omega
\end{aligned}
$$

It is worth noting that $sgn(\omega)$ is an odd function and hence $sgn(\omega)\sin\omega t$ is an even function, and $sgn(\omega)\cos\omega t$ is an odd function. Hence the sine function will survive.

Remark

Another method to calculate the Hilbert transform of the Hermite function (Gaussian pulse) $\pi^{-\frac{1}{4}} e^{-t^2/2}$ is to multiply the Hilbert transformer $(-isgn(\omega))$ by the spectrum of the Hermite function and use the inverse Fourier transform. No infinite integral is needed in the calculations of the Hermite functions. Therefore, the error does not propagate.

A little note, as we have already seen before, that the Hilbert transform of the delta pulse $\delta(t)$ gives us the Hilbert transformer $\frac{1}{\pi t}$ and the Fourier transform of the

Hilbert transformer gives us the sign shift function (signum) $(-isgn(\omega))$, that is

$$\mathcal{H}\{\delta(t)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} P.V. \frac{\delta(\tau)}{t-\tau} d\tau = \frac{1}{\pi t}$$

$$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = (-isgn(\omega)).$$

This information is very important in our study of the Hilbert transform and the related Fourier transform. We next turn our attention to the study of finite Hilbert transform.

5.14 The finite Hilbert transform

We shall now study equations with Cauchy's principal integrals over a finite interval. These equations have important applications, for example, in aerodynamics.

Of fundamental importance in this study is the finite Hilbert transform

$$f(x) = \frac{1}{\pi} P.V. \int_{-1}^{1} \frac{\phi(t)}{t-x} dt, \tag{5.54}$$

where we assume the basic interval $(-1, 1)$. Until recently this transformation, in contrast to the infinite one, has received little attention. A few of its properties can be deduced from the corresponding properties of the infinite transformations, by supposing that the function ϕ vanishes identically outside the interval $(-1, 1)$.

For instance, from Parseval's theorem 5.2, we obtain

$$\int_{-1}^{1} \{\phi_1(x)\mathcal{H}_x\{\phi_2(t)\} + \phi_2(x)\mathcal{H}_x\{\phi_1(t)\}\} \, dx = 0, \tag{5.55}$$

provided that the functions $\phi_1(x)$ and $\phi_2(x)$ belong to the classes L_{p_1} and L_{p_2}, respectively, in the basic interval $(-1, 1)$, and that

$$\frac{1}{p_1} + \frac{1}{p_2} \leq 1. \tag{5.56}$$

Similarly, if $\frac{1}{p_1} + \frac{1}{p_2} < 1$, we obtain from Theorem 5.4

$$\mathcal{H}[\phi_1\mathcal{H}\{\phi_2\} + \phi_2\mathcal{H}\{\phi_1\}] = \mathcal{H}\{\phi_1\}\mathcal{H}\{\phi_2\} - \phi_1\phi_2. \tag{5.57}$$

In other case, however, the transformation $\mathcal{H}$ requires special treatment. For inversion formula, in L_p space

$$\phi(x) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt, \tag{5.58}$$

which can be immediately deduced from Theorem 5.1, is not satisfactory because its use requires knowledge of the function $f(x)$ outside of the basic interval

$(-1, 1)$, where it generally does not vanish. One difficulty in the study of the finite Hilbert transform is that there exists no simple reciprocity theorem like Theorem 5.1; in fact, the transformation has no unique inverse in the L_p-space $(p > 1)$. For more information the reader is referred to Tricomi [5].

For instance, the finite Hilbert transform of the function $(1-x^2)^{-\frac{1}{2}}$, which belongs to the class 2-0, vanishes identically in the basic interval $(-1, 1)$; for if we put $y = \frac{(1-t^2)}{(1+t^2)}$, we find

$$\begin{aligned}
\mathcal{H}_x[(1-y^2)^{-\frac{1}{2}}] &= \frac{1}{\pi} P.V. \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} \frac{dy}{y-x} \\
&= \frac{2}{\pi} P.V. \int_{0}^{\infty} \frac{dt}{(1-x)-(1+x)t^2} \\
&= \frac{1}{\pi\sqrt{1-x^2}} \left[\ln \left| \frac{\sqrt{1-x}+\sqrt{1+x}t}{\sqrt{1-x}-\sqrt{1+x}t} \right| \right]_0^{\infty} = 0. \qquad (5.59)
\end{aligned}$$

However, outside of the interval $(-1, 1)$, we have

$$\begin{aligned}
\mathcal{H}_x[(1-y^2)^{-\frac{1}{2}}] &= \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} \frac{dy}{y-x} \\
&= -\frac{2}{\pi} \int_{0}^{\infty} \frac{dt}{(x-1)+(x+1)t^2} \\
&= -\frac{2}{\pi\sqrt{x^2-1}} \left[\tan^{-1} \left(\frac{\sqrt{x+1}t}{\sqrt{x-1}} \right) \right]_0^{\infty} \\
&= -\frac{1}{\sqrt{x^2-1}}. \qquad (5.60)
\end{aligned}$$

Note that as a consequence of equation (5.59), we obtain

$$\begin{aligned}
\mathcal{H}_x[(1-y^2)^{\frac{1}{2}}] &= \frac{1}{\pi} P.V. \int_{-1}^{1} \frac{1-y^2}{\sqrt{1-y^2}} \frac{dy}{y-x} \\
&= -\frac{1}{\pi} P.V. \int_{-1}^{1} \frac{y^2}{\sqrt{1-y^2}} \frac{dy}{y-x} \\
&= -\frac{1}{\pi} P.V. \int_{-1}^{1} \frac{y^2-x^2+x^2}{\sqrt{1-y^2}} \frac{dy}{y-x} \\
&= -\frac{1}{\pi} P.V. \int_{-1}^{1} \frac{y+x}{\sqrt{1-y^2}} dy \\
&= -\frac{x}{\pi} P.V. \int_{-1}^{1} \frac{dy}{\sqrt{1-y^2}} \\
&= -x. \qquad (5.61)
\end{aligned}$$

The main problem for the finite Hilbert transform is to find its inversion formula in the L_p-space ($p > 1$), that is to solve the **airfoil equation (5.54)** by means of L_p-function ($p > 1$). It must, of course, be assumed that the given function $f(x)$ itself belongs to the class L_p.

5.14.1 Inversion formula for the finite Hilbert transform

To find an inversion formula we can use the convolution Theorem 5.4, which can be applied to the function pair

$$\phi_1(x) \equiv \phi(x), \quad \phi_2(x) \equiv (1 - x^2)^{\frac{1}{2}},$$

because the second function, being obtained, belongs to any class, even for very large p_2. We thus obtain the equality

$$\mathcal{H}_x[-y\phi(y) + \sqrt{1 - y^2} f(y)] = -xf(x) - \sqrt{1 - x^2}\phi(x). \tag{5.62}$$

On the other hand, we have

$$\mathcal{H}_x[y\phi(y)] = \frac{1}{\pi} P.V. \int_{-1}^{1} \frac{y - x + x}{y - x}\phi(y)dy = \frac{1}{\pi}\int_{-1}^{1} \phi(y)dy + xf(x).$$

Hence from equation (5.62) it follows that necessarily

$$-\frac{1}{\pi}\int_{-1}^{1} \phi(y)dy + \mathcal{H}_x\left[\sqrt{1 - y^2} f(y)\right] = -\sqrt{1 - x^2}\phi(x),$$

that is

$$\sqrt{1 - x^2}\phi(x) = -\mathcal{H}_x\left[\sqrt{1 - y^2} f(y)\right] + C, \tag{5.63}$$

or, more explicitly,

$$\phi(x) = -\frac{1}{\pi} P.V. \int_{-1}^{1} \sqrt{\left(\frac{1 - y^2}{1 - x^2}\right)} \frac{f(y)}{y - x} dy + \frac{C}{\sqrt{1 - x^2}}. \tag{5.64}$$

Here, in view of equation (5.59) the constant

$$C = \frac{1}{\pi}\int_{-1}^{1} \phi(y)dy \tag{5.65}$$

has the character of an arbitrary constant.

The significance of the previous result is the following: if the given equation (5.54) has any solution at all of class L_p ($p > 1$), then this solution must have the

form as in equation (5.64). Consequently, the only nontrivial solutions of the class L_p ($p > 1$) of the homogeneous equation

$$\frac{1}{\pi} P.V. \int_{-1}^{1} \frac{\phi(y)}{y-x} dy = 0 \tag{5.66}$$

are $C(1-x^2)^{-\frac{1}{2}}$.

In view of the identities

$$\begin{aligned}
\sqrt{\left(\frac{1-y^2}{1-x^2}\right)} &= \sqrt{\left(\frac{1+x}{1-x}\right)}\sqrt{\left(\frac{1-y}{1+y}\right)}\left(1+\frac{y-x}{1+x}\right) \\
&= \sqrt{\left(\frac{1-x}{1+x}\right)}\sqrt{\left(\frac{1+y}{1-y}\right)}\left(1-\frac{y-x}{1-x}\right)
\end{aligned}$$

solution (5.64) can be put into the two further alternative forms

$$\begin{aligned}
\phi(x) &= -\frac{1}{\pi}\sqrt{\left(\frac{1+x}{1-x}\right)} P.V. \int_{-1}^{1} \sqrt{\left(\frac{1-y}{1+y}\right)} \frac{f(y)}{y-x} dy + \frac{C_1}{\sqrt{1-x^2}} \\
\phi(x) &= -\frac{1}{\pi}\sqrt{\left(\frac{1-x}{1+x}\right)} P.V. \int_{-1}^{1} \sqrt{\left(\frac{1+y}{1-y}\right)} \frac{f(y)}{y-x} dy + \frac{C_2}{\sqrt{1-x^2}}
\end{aligned} \tag{5.67}$$

5.14.2 Trigonometric series form

Some authors use trigonometric series form to solve the airfoil equation. This method is theoretically less satisfying than the present one; however, it may be useful in practice.

Using the finite Hilbert transform we can prove the following identities:

$$P.V. \int_0^{\pi} \frac{\cos(n\eta)}{\cos\eta - \cos\xi} d\eta = \pi \frac{\sin(n\xi)}{\sin\xi} \qquad (n = 0, 1, 2, 3, \ldots) \tag{5.68}$$

$$P.V. \int_0^{\pi} \frac{\sin((n+1)\eta)\sin\eta}{\cos\eta - \cos\xi} d\eta = -\pi\cos(n+1)\xi, \tag{5.69}$$

These pair of results show that the finite Hilbert transform operates in a particularly simple manner on the Tchebichef polynomials

$$T_n(\cos\xi) = \cos(n\xi), U_n(\cos\xi) = \frac{\sin(n+1)\xi}{\sin\xi}.$$

To be precise, we have the finite Hilbert transform

$$\mathcal{H}\{(1-t^2)^{-\frac{1}{2}} T_n(t)\} = U_{n-1}(x) \qquad (n = 1, 2, 3, \ldots) \tag{5.70}$$

Hence, by expanding $f(x)$ in a series of polynomials $U_n(x)$, we can immediately deduce (at least formally) a corresponding expansion of $\phi(x)$ in series of polynomials $T_n(t)$, if we neglect the factor $(1-t^2)^{-\frac{1}{2}}$. Formula (5.70), as well as the similar one

$$\mathcal{H}_x[(1-t^2)^{\frac{1}{2}}U_{n-1}(t)] = -T_n(x), \tag{5.71}$$

can be readily proved with the help of Theorem 5.3, by starting with the analytic functions

$$\Phi(z) = -(1-z)^{-\frac{1}{2}}\left[z-\sqrt{(1-z^2)}\right]^n \quad \text{and}$$

$$\Phi(z) = \left[z-\sqrt{(1-z^2)}i\right]^n,$$

respectively.

5.14.3 An important formula

Using the same method we can also prove the important formula

$$\mathcal{H}_x\left[\left(\frac{1-t}{1+t}\right)^{\alpha}\right] = \cot g(\alpha\pi)\left(\frac{1-x}{1+x}\right)^{\alpha} - \frac{1}{\sin(\alpha\pi)}, \quad (0<|\alpha|<1). \tag{5.72}$$

We start with the analytic function

$$\Phi(z) = \left(\frac{z-1}{z+1}\right)^{\alpha} - 1, \tag{5.73}$$

which satisfies the condition $\int_{-\infty}^{\infty}|\Phi(x+iy)|^p dx < K$ for $(p>1)$, where K is a positive constant. Because of $|z+1|>2$ we have

$$\Phi(z) = -\alpha\frac{2}{z+1} + \binom{\alpha}{2}\left(\frac{2}{z+1}\right)^2 - \binom{\alpha}{3}\left(\frac{2}{z+1}\right)^3 + \cdots.$$

Equation (5.72) is then an immediate consequence of Theorem 5.3, because on the real axis the function, i.e. equation (5.73) reduces to a function $\Phi(x+i0)$ which is real outside of $(-1,1)$, and for $-1<x<1$ we have

$$\begin{aligned}\Phi(x+i0) &= \left(\frac{1-x}{1+x}\right)^{\alpha} e^{i\alpha\pi} - 1\\ &= \left(\frac{1-x}{1+x}\right)^{\alpha}\cos(\pi\alpha) - 1 + i\left(\frac{1-x}{1+x}\right)^{\alpha}\sin(\pi\alpha).\end{aligned} \tag{5.74}$$

Furthermore, equation (5.72) is interesting because it shows that in some cases a function $\phi(y)$ which becomes infinite like $A(1-y)^{-\alpha}$ or $A(1+y)^{-\alpha}$ $0<\alpha<1$

as $y \to \pm 1$ is carried by the finite Hilbert transform into a function with similar behaviour, if we neglect the fact that A is replaced by $\pm A \cot g(\pi\alpha)$. That this behaviour is common and shown by Tricomi [5] using asymptotic theorem.

Remark

Some explanation with regard to the result (5.72) is given below.

$$Re\Phi(x + i0) = \left(\frac{1-x}{1+x}\right)^{\alpha} \cos(\pi\alpha) - 1$$

$$Im\Phi(x + i0) = \left(\frac{1-x}{1+x}\right)^{\alpha} \sin(\pi\alpha)$$

$$\mathcal{H}_x[Im\Phi(x + i0)] = Re\Phi(x + i0)$$

$$\frac{1}{\pi}\int_{-1}^{1}\left(\frac{1-y}{1+y}\right)^{\alpha} \frac{\sin(\pi\alpha)}{y-x}dy = \left(\frac{1-x}{1+x}\right)^{\alpha} \cos(\pi\alpha) - 1$$

$$\frac{1}{\pi}\int_{-1}^{1}\left(\frac{1-y}{1+y}\right)^{\alpha} \frac{dy}{y-x} = \left(\frac{1-x}{1+x}\right)^{\alpha} \cot(\pi\alpha) - \frac{1}{\sin(\pi\alpha)}$$

Hence the result (5.72) follows.

5.15 Sturm–Liouville problems

Variational methods can be used to study the solution procedures of differential equations. We have in the meantime observed that the problem of extremizing an integral leads to one or more differential equations. Let us turn our attention to study the boundary value problems by using the variational calculus. Although this cannot always be done, it is possible to investigate in some very important cases. By way of illustrating, we obtain with the general second-order Sturm–Liouville equation

$$[r(x)y']' + [q(x) + \lambda p(x)]\, y = 0, \tag{5.75}$$

where the functions r, p, and q are continuous, $r(x) \neq 0$ and $p(x) > 0$ on a fundamental interval $[a, b]$. Multiplying equation (5.75) by y throughout, and then integrating from a to b, and subsequently solving for λ algebraically we have

$$\lambda = \frac{-\int_a^b \{q(x)y^2 + y[r(x)y']'\}dx}{\int_a^b p(x)y^2 dx} = \frac{I}{J}. \tag{5.76}$$

Now, if y is a solution of equation (5.75), cy is a solution also, where c is an arbitrary constant. In fact, since $p(x) > 0$ on $[a, b]$, every solution of equation (5.75) is expressed in terms of solutions that satisfy the normalized condition.

$$J = \int_a^b p(x)y^2 dx = 1. \tag{5.77}$$

Imposing this constraint, the last equation in equation (5.76) becomes

$$-\int_a^b \{q(x)y^2 + y[r(x)y']'\}dx = I,$$

or, integrating $y[r(x)y']'$ by parts,

$$I = \int_a^b (r(x)y'^2 - q(x)y^2)dx + r(a)y(a)y'(a) - r(b)y(b)y'(b). \tag{5.78}$$

For a Sturm–Liouville problem consisting of equation (5.75) and a pair of fixed end-point conditions

$$\left.\begin{aligned} y(a) &= 0 \\ y(b) &= 0 \end{aligned}\right\}. \tag{5.79}$$

The last two terms in equation (5.78) vanish leaving

$$I = \int_a^b (r(x)y'^2(x) - q(x)y^2)dx. \tag{5.80}$$

The problem of reducing this integral stationary, subject to the constraint in equation (5.77), and relative to functions that vanish at $x = a$ and $x = b$, is clearly an isoperimetric problem, Its corresponding Euler−Lagrange equation is

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0,$$

where $f^* = r(x)(y')^2 - q(x)y^2 - \lambda p(x)y^2$. It is easily verified that these relations yield equation (5.75).

$$\left[\begin{aligned} &\frac{\partial f^*}{\partial y} = -2q(x)y - 2\lambda p(x)y \\ &\frac{\partial f^*}{\partial y'} = 2y(x)y' \quad \textit{and} \quad \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 2(r(x)y')' \\ &\textit{and} \quad \textit{so} \quad (r(x)y')' + (q(x) + \lambda p(x)y)y = 0 \end{aligned}\right].$$

Thus, the functions that solve our isoperimetric problem must be normalized solutions of the Sturm−Liouville problem consisting of equations (5.75) and (5.79), the normalized being with respect to $p(x)$. Of course, these solutions are just the normalized characteristic functions $y_1, y_2 \ldots y_n$ that correspond to the characteristic rules of the Sturm−Liouville problem. With the characteristic values $\lambda_1, \lambda_2 \ldots \lambda_k$ arranging in increasing order, it is possible to show that λ_k is the minimum of

the integral equation (5.80) relative to the suitable differentiable functions y that satisfies equations (5.79), (5.77), and the $k-1$ orthogonality relations

$$\int_a^b p(x)y_i y dx = 0 \qquad i = 1, 2, \ldots k-1.$$

The I of equation (5.80) that takes on the values of λ_k when y is replaced by y_k can be established as follows. First, replace y by y_k in the integral of I. Then integrate the term $r(x)(y_k')^2$ by parts, observing that $y_k(a) = y_b(a) = 0$. The result of all this is

$$\begin{aligned}
I &= \int_a^b [r(x)\{y_k'\}^2 - q(x)y_k^2]dx \\
&= [r(x)y'(x) \int y_k' dx]_a^b - \int_a^b y_k\{r(x)y_k'\}' dx - \int_a^b q(x)y_k^2 dx \\
&= [r(x)y_k'(x)y_k(x)]_a^b - \int_a^b [y_k\{r(x)y_k'(x)\}' + q(x)y_k^2]dx \\
&= -\int_a^b y_k\{\{r(x)y_k'\}' + q(x)y_k\}dx.
\end{aligned}$$

Since $[r(x)y_k']' + q(x)y_k(x)] = -\lambda_k p(x)y_k$ and $\int_a^b p(x)y_k^2 dx = 1$, we have

$$I = \int_a^b \lambda_k y_k^2 p(x)dx = \lambda_k.$$

Summary of Sturm–Liouville problems

A Sturm–Liouville problem consisting of equation (5.75) and two free end-point conditions

$$\left.\begin{aligned} a_1 r(a)y'(a) + a_2 y(a) = 0 \\ b_1 r(b)y'(b) + b_2 y(b) = 0 \end{aligned}\right\} \tag{5.81}$$

can also be related to an isoperimetric problem. In this case, we may set $h = a_2/a_1$ and $k = b_2/b_1$ and utilize equation (5.81) to write equation (5.78) as

$$I = \int_a^b [r(x)(y')^2 - q(x)y^2]dx + ky^2(b) - hy^2(a). \tag{5.82}$$

The last two terms of this equation can be incorporated into the integral by introducing any continuously differentiable function $g(x)$ on $[a, b]$ for which $g(a) = h$ and $g(b) = k$. Indeed, this enables us to express equation (5.82) as

$$I = \int_a^b \left[r(x)(y')^2 - q(x)y^2 + \frac{d}{dx}(gy^2) \right] dx. \tag{5.83}$$

The isoperimetric problem requires that this integral be rendered stationary, subject to equation (5.77), and relative to continuously differentiable function defined on $[a,b]$. Also it has equation (5.75) as its Euler–Lagrange equation because $\frac{d}{dx}(gy^2)$ is an exact derivative. This can, of course, be verified directly. Hence the choice of g has no effect on the stationary values of I. Since no conditions have been imposed on the comparison functions at the end-points of the specified interval $[a, b]$, a natural boundary condition

$$\frac{\partial f^*}{\partial y'} = 0$$

where

$$f^* = \left[r(x)(y')^2 - q(x)y^2 + \frac{d}{dx}(gy^2)\right] - \lambda p(x)y^2,$$

must hold at $x=a$ and at $x=b$. Differentiating f^* partially with respect to y', we find

$$\begin{aligned}\frac{\partial f^*}{\partial y'} &= \frac{\partial}{\partial y'}\left[r(x)y'^2 - q(x)y^2 + \frac{d}{dx}(gy^2)\right]\\ &= 2r(x)y' + 2gy\end{aligned}$$

and so the condition $r(x)y' + gy = 0$ must hold at each end of $[a, b]$. Substituting $x=a$ and $x=b$, in turn, into this equation we get equation (5.81). The choice of g is again of no consequence so long as $g(a) = a_2/a_1$ and $g(b) = b_2/b_1$.

Example 5.8: A practical application

In many practical applications, a variational formulation of the problem being investigated is easy to derive. To illustrate, we shall find an isoperimetric problem that when solved yields the natural frequency of a string of weight $w(x)$ stretched under tension T between $x=0$ and $x=l$ and vibrating transversely about a horizontal x-axis in the xy plane. If the change in length of the string during its motion is so small that T can be assumed constant, the potential energy stored in the string by virtue of the work done against T is equal to T times the change of length in the string:

$$P.E. = T\int_0^l \left[\sqrt{1+y'^2} - 1\right] dx.$$

Here, $ds = \sqrt{dx^2 + dy^2} = \sqrt{1+y'^2}dx$ and $ds - dx$ is the change of the elementary length. If, furthermore, the variation is such that $|y'| \ll 1$, then by expanding $\sqrt{1+y'^2}$ by the binomial expansion and retaining only the dominant term in the integrand, we have

Some mathematical formulas

By binomial expansion,

$$
\begin{aligned}
(1+y'^2)^{\frac{1}{2}} &= 1+\frac{1}{2}y'^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(y'^2)^{\frac{1}{2}-1}}{2!}+\cdots \\
(1+x)^m &= 1+C_1^m x+C_2^m x^2+\cdots \\
(1+x)^{\frac{1}{2}} &= 1+C_1^{\frac{1}{2}}x+C_2^{\frac{1}{2}}x^2+\cdots \\
&= 1+\frac{x}{2}-\frac{1}{8}x^2+\cdots \\
(1+x)^{-\frac{1}{2}} &= 1-\frac{x}{2}+\frac{3}{8}x^2-\frac{5}{16}x^3+\cdots
\end{aligned}
$$

The potential energy stored in the string by virtue of its elongation can be expressed as

$$\frac{T}{2}\int_0^l (y')^2 dx.$$

If the string is also subjected to a disturbed force of magnitude per unit length in the x direction $|f(x)y|$ acting vertically [for example, an elastic restoring force $(-ky)$], then the string has additional potential energy given by

$$-\int_0^l \int_0^y f(x)sdsdx = -\int_0^l \frac{1}{2}f(x)y^2 dx,$$

which is the work required due to the disturbed force, to change the deflection curve of the string from the segment $[0, l]$ of the x-axis into the curve determined by y. Thus, the total potential energy of the string is

$$\frac{1}{2}\int_0^l [T(y')^2 - f(x)y^2]dx.$$

Similarly, the total instantaneous kinetic energy of the string is

$$\frac{1}{2}\int_0^l \frac{w(x)}{g}\dot{y}^2 dx.$$

The displacement y that we have been considering is actually a function of x and t of the form $y = X(x)\cos \omega t$. Hence, substituting into the two energy expressions, and applying the principle that during free vibrations the maximum value of the kinetic energy is equal to the maximum value of the potential energy, we obtain

$$(K.E.)_{max} = (P.E.)_{max}$$

$$\frac{1}{2}\int_0^l \frac{w(x)}{g}(X^2(-\omega)^2\sin^2\omega t)dx = \frac{1}{2}\int_0^l [T(X'\cos\omega t)^2 - f(x)X^2\cos^2\omega t]dx$$

$$\omega^2 = \frac{\int_0^l [T(X')^2 - f(x)X^2]dx}{\int_0^l \left(\frac{w(x)}{g}\right)X^2 dx},$$

where $|\cos^2\omega t| = 1$ and $|\sin^2\omega t| = 1$. This equation is satisfied by a nontrivial function X if and only if it is satisfied by cX, where c is a nonzero parameter. Thus, we need consider only function X that is normal with respect to $\frac{w(x)}{g}$. The problem of extremizing the integral

$$I = \int_0^l [T(X')^2 - f(x)X^2]dx,$$

subject to the constraint

$$J = \int_0^l \left(\frac{w(x)}{g}\right)X^2 dx = 1,$$

and relative to the appropriate comparison functions, is obviously an example of an isoperimetric problem like those just discussed. With

$$f^* = [T(X')^2 - f(x)X^2] - \omega^2\left[\frac{w(x)}{g}\right]X^2$$

we find, in the usual way, that the corresponding Euler–Lagrange equation is

$$[TX']' + \left[f(x) + \omega^2\left\{\frac{w(x)}{g}\right\}\right]X = 0 \tag{5.84}$$

$$\frac{d}{dx}\left[\frac{df^*}{dX'}\right] - \frac{\partial f^*}{\partial X} = 0$$

$$[T(2X')]' + 2f(x)X + \omega^2\left(\frac{w(x)}{g}\right)(2X) = 0.$$

Therefore, after some reduction we obtain $(TX')' + [f(x) + \omega^2(\frac{w(x)}{g})]X = 0$. This is precisely the equation for the space factor X that results when the partial differential equation governing the vibration of a nonuniform string is solved by the method of separation of variables. If both ends of the string are fixed on the x-axis, so that

$$y(0,t) = y(l,t) = 0 \quad \text{for all } t,$$

then all comparison functions of the isoperimetric problem, and in particular, the solutions of equation (5.84) must satisfy the fixed-end conditions $X(0)=X(l)=0$. On the other hand, if at both ends of the string the displacement y is unspecified, then all comparison functions, and in particular the solutions of equation (5.84), must satisfy natural boundary conditions of the form

$$\frac{\partial f^*}{\partial X'} = 2TX' = 0 \ \ at \ \ x = 0 \ \ and \ \ at \ x = l,$$

given $2T \neq 0$, these reduce to the free-end conditions

$$X'(0) = 0 \quad and \quad x'(l) = 0.$$

More general free-end conditions of the type

$$\left.\begin{aligned} &TX'(0) + hX(0) = 0 \quad and \\ &TX'(l) + hX(l) = 0 \end{aligned}\right\} \tag{5.85}$$

might also apply. Such conditions, arise, for instance, when each end of the string is restrained by a nonzero restoring force ϕ proportional to the displacement, say $\phi(0)=ay_0$ and $\phi(l)=by_l$. The potential energy stored in the system by virtue of the work done against these forces is

$$\tfrac{1}{2}ay_0^2 + \tfrac{1}{2}by_l^2,$$

and hence the total potential energy of the system is now

$$\frac{1}{2}\int_0^l [T(y')^2 - f(x)y^2]dx + \frac{1}{2}ay_0^2 + \frac{1}{2}by_l^2.$$

To incorporate the last two terms into the integral, let $g(x)$ be an arbitrary differentiable function of x such that

$$g(0) = -a \quad and \quad g(l) = b. \quad Then$$

$$\begin{aligned} \textstyle\int_0^l \frac{d}{dx}[g(x)y^2]dx &= g(x)y^2|_0^l \\ &= g(l)y_l^2 - g(0)y_0^2 \\ &= by_l^2 + ay_0^2. \end{aligned}$$

Hence the expression for the instantaneous potential energy of the string can be rewritten in the form

$$\frac{1}{2}\int_0^l \left\{T(y')^2 - f(x)y^2 + \frac{d}{dx}(g(x)y^2)\right\} dx.$$

Again setting $y = X(x)\cos\omega t$ and equating the maximum values of the potential energy and the kinetic energy, we are led to an isoperimetric problem in which

$$I = \int_0^l \left\{ T(X')^2 - f(x)X^2 + \frac{d}{dx}(g(x)X^2) \right\} dx,$$

while J is unchanged. With

$$f^* = \{T(X')^2 - f(x)X^2 + g'(x)X^2 + 2g(x)XX'\} - \omega^2[w(x)/g]X^2$$

we find the natural boundary conditions

$$\frac{\partial f^*}{\partial x'} = 0 \quad \textit{giving}$$
$$TX' + gX = 0,$$

which, when evaluated at $x = 0$ and at $x = l$, yields conditions on x of general-form solution in equation (5.84).

Example 5.9

Let us finally consider a general nonhomogeneous second-order linear differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = \phi(x),$$

which is normal on an interval $[a, b]$. We know all such equations can be written in the form

$$[r(x)y']' + q(x)y = W(x). \tag{5.86}$$

This equation will be the Euler–Lagrange equation for an integral of the type

$$I = \int_a^b f(x, y, y')dx,$$

if $\frac{\partial f}{\partial dy'} = r(x)y'$ and $\frac{\partial f}{\partial dy} = W(x) - q(x)y$. From the last two relations we get

$$f = \tfrac{1}{2}r(x)(y')^2 + u(x, y) \quad \textit{and}$$
$$f = W(x)y - \tfrac{1}{2}q(x)y^2 + v(x, y').$$

Sufficient conditions for these two representations of f to be identical are that

$$u(x, y) = W(x)y - \tfrac{1}{2}q(x)y^2 \quad \textit{and}$$
$$v(x, y') = \tfrac{1}{2}r(x)(y')^2,$$

in which case I becomes the variational integral

$$I = \int_a^b \left[\frac{1}{2}r(x)(y')^2 - \frac{1}{2}q(x)y^2 + W(x)y\right] dx. \tag{5.87}$$

It is a simple matter to verify that equation (5.86) is the Euler–Lagrange equation for this integral. In fact, if an exact derivative $\frac{d}{dx}[g(x,y)]$ is added to the integral of equation (5.87), equation (5.86) is still the corresponding Euler–Lagrange equation. By defining f^* as usual, with $-\lambda/2$ as the Lagrange multiplier, it is easy to show that the isoperimetric problem involving the integral equation (5.87) and an integral constraint

$$J = \int_a^b p(x)y^2 dx, \ J \ a \ constant, \tag{5.88}$$

and $p(x) \neq 0$ on $[a,b]$ has as its Euler–Lagrange equation,

$$[ry']' + [q(x) + \lambda p(x)]y = W(x). \tag{5.89}$$

Of course, equations (5.86) and (5.89)will be accompanied by prescribed or natural boundary conditions, describing on how comparison functions of the related variational problems behave at the end points of $[a,b]$.

5.16 Principles of variations

A real-valued function f whose domain is a set of real functions $\{y\}$ is sometimes called a **functional** or, more specifically, a **functional of a single independent variable**. Functionals of several independent variables are also of interest. With ordinary functions, the values of independent variables are numbers. However, with functional variables, the values of the independent variables are functionals. In general, the value of a function changes when the values of its independent variables change. So, we need to estimate the value of changes of a functional. To see how this could be done, let us consider a function $F(x,y,y')$ that, when x is held fixed, becomes a functional defined on a set of functions $\{y\}$, and let us develop an estimate for the change in F corresponding to an assigned change in the value of $y(x)$ of a function y in $\{y\}$ for a fixed value of x. If $y(x)$ is changed into

$$y(x) + \epsilon\eta(x),$$

where ϵ is independent of x we call the change $\epsilon\eta(x)$, the **variation** of $y(x)$ and denote it by

$$\delta y = \epsilon\eta(x). \tag{5.90}$$

Moreover, from the changed value of y we define that the changed value of $y'(x)$ is

$$y'(x) + \epsilon\eta(x).$$

Hence we have the comparison formula

$$\delta y'(x) = \epsilon\eta'(x) \tag{5.91}$$

for the variation of $y'(x)$. Corresponding to these changes we have the change

$$\Delta F = F(x, y + \epsilon\eta, y' + \epsilon\eta') - F(x, y, y').$$

If we expand the first term on the right in a MacLaurin's expression in powers of ϵ, we have

$$\Delta F = F(x, y, y') + \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)\epsilon + \left(\frac{\partial^2 F}{\partial y^2}\eta^2 + 2\frac{\partial^2 F}{\partial y \partial y'}\eta' + \frac{\partial^2 F}{\partial y'^2}\eta'^2\right)\frac{\epsilon^2}{2!}$$
$$+ \cdots - F(x, y, y')$$

or, neglecting powers of ϵ higher than the first,

$$\Delta F \doteq \frac{\partial F}{\partial y}(\epsilon\eta) + \frac{\partial F}{\partial y'}(\epsilon\eta')$$
$$\Delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'.$$

By analogy with the differential of a function, we define the last expression to be the **variation** of the functional F and denote it by δF.

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'. \tag{5.92}$$

Remark

By strict analogy with the differential of a function of three variables, we must have expressed the definition $\delta F = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$ if x is a variable parameter. It is worth noting that in its simplest form the differential of a function is a first-order approximation to the change in the function as x varies along a particular curve, whereas the variation of a functional is a first-order approximation to the change in the functional at a particular value of x as we move from curve to curve. It is interesting and important to note that variations can be calculated by the same rules that apply to differentials. Specifically

$$\delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2 \tag{5.93}$$

$$\delta(F_1 F_2) = F_1 \delta F_2 + F_2 \delta F_1 \tag{5.94}$$

$$\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2 \delta F_1 - F_1 \delta F_2}{F_2^2} \tag{5.95}$$

$$\delta(F^n) = nF^{n-1}\delta F. \tag{5.96}$$

These relations are easy to prove by means of equation (5.92). For example, with F replaced by F_1F_2, equation (5.92) yields

$$\begin{aligned}
\delta(F_1F_2) &= \frac{\partial}{\partial y}(F_1F_2)\delta y + \frac{\partial}{\partial y'}(F_1F_2)\delta y' \\
&= \left[F_1\left(\frac{\partial F_2}{\partial y}\right) + F_2\left(\frac{\partial F_1}{\partial y}\right)\right]\delta y + \left[F_1\left(\frac{\partial F_2}{\partial y'}\right) + F_2\left(\frac{\partial F_1}{\partial y'}\right)\right]\delta y' \\
&= F_1\left[\frac{\partial F_2}{\partial y}\delta y + \frac{\partial F_2}{\partial y'}\delta y'\right] + F_2\left[\frac{\partial F_1}{\partial y}\delta y + \frac{\partial F_1}{\partial y'}\delta y'\right] \\
&= F_1\delta F_2 + F_2\delta F_1,
\end{aligned}$$

as asserted by equation (5.94). Proofs of equations (5.93), (5.95), and (5.96) can be performed following the above steps. From the definite relations, i.e. equations (5.90) and (5.16), and with $D = \frac{d}{dx}$, we have

$$\delta Dy = \delta y' = \epsilon\eta' = \epsilon D\eta = D(\epsilon\eta) = D(\delta y),$$

and hence δ and D commute; that is, taking the variation of a function $y(x)$, and differentiating it with respect to its **independent variables** are commutative operations. We can, of course, consider functionals of more than one function, and the variations of such functionals are defined by expressions analogous to equation (5.92). For instance, for the functional $F(x, u, v, u', v')$ we have

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial v'}\delta v'.$$

Similarly, we can consider variations of functionals that depend on functions of more than one variable. The functional $F(x, y, u, u_x, u_y)$, for example, whose value depends, for fixed x and y, on the function $u(x, y)$, we have

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y. \tag{5.97}$$

For a functional expressed as a definite integral, say the integral

$$I(y) = \int_a^b f(x, y, y')dx,$$

of the kind we discussed already, we have, first

$$\Delta I = I(y + \epsilon\eta) - I(y).$$

If the limits of I do not depend on y, we have furthermore,

$$\begin{aligned}\Delta I &= \int_a^b f(x,y+\epsilon\eta,y'+\epsilon\eta')dx - \int_a^b f(x,y,y')dx\\ &= \int_a^b [f(x,y+\epsilon\eta,y'+\epsilon\eta') - f(x,y,y')]dx\\ &= \int_a^b \Delta f(x,y,y')dx.\end{aligned}$$

The variation of I is now defined as the expression resulting when Δf in the last integral is replaced by the first-order approximation δf; that is

$$\delta I = \int_a^b \delta f(x,y,y')dx. \tag{5.98}$$

Writing equation (5.98) as

$$\delta \int_a^b f(x,y,y')dx = \int_a^b \delta f(x,y,y')dx,$$

we see that integrating a functional $f(x,y,y')$ over an interval $[a,b]$ and taking the variation of $f(x,y,y')$ are commutative operators, i.e. the operator symbolized by

$$\int_a^b (.)dx \quad and \quad \delta\ commute.$$

From calculus we recall that a necessary condition for a function to have an extremum is that its *differential* vanishes. We can now show, similarly, that a necessary condition for a functional I to have an extremum is that its *variations* vanish. In fact, using the results of the preceding discussion, we write

$$\begin{aligned}\delta I &= \int_a^b \delta f(x,y,y')dx\\ &= \int_a^b (f_y\delta y + f_{y'}\delta y')dx\\ &= \int_a^b \left[f_y\delta y + f_{y'}\frac{d}{dx}(\delta y)\right]dx.\end{aligned}$$

Now, integrating the last term by parts, we obtain

$$\int_a^b \left(f_{y'}\frac{d}{dx}(\delta y)\right)dx = f_{y'}(\delta y)|_a^b - \int_a^b \frac{d}{dx}(f_{y'})\delta y dx,$$

when we assume that at $x=a$ and at $x=b$ either the variation $\delta y = \epsilon\eta(x)$ is zero because $\eta(x)$ is, or a natural boundary condition holds so that $F_{y'}$ vanishes, it follows that the integral portion of the last equation is equal to zero. Hence we have

$$\delta I = \int_a^b \left[f_y - \frac{d}{dx}(f_{y'}) \right] \delta y dx,$$

since we have already seen that $f_y - \frac{d}{dx}(f_{y'}) = 0$ is a necessary condition for an extremum of I, it follows that δI is also zero at any extremum of I. Conversely, since δy is an arbitrary variation in y, the condition $\delta I = 0$ implies that

$$f_y - \frac{d}{dx}[f_{y'}] = 0,$$

which is the Euler–Lagrange equation.

5.17 Hamilton's principles

Although Newton's law of motion forms the basic foundation for the investigation of mechanical phenomena still over the years there have been refinements and extensions of his law that often provide more effective methods to study the applied problems. In this section, we will take a brief look at two of these,

(a) Hamilton's principle and
(b) Lagrange's equations

(a) Hamilton's principle

Let us consider a mass particle moving in a force field $\mathbf{F}$. Let $\mathbf{r}(t)$ be the position vector from the origin to the instantaneous position of the particle. Then, according to Newton's second law in vector form, the actual path of the particle is described by the equation

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}. \tag{5.99}$$

Now, consider any other path joining the points where the particle is located at $t = t_1$ and at $t = t_2$. Such a path is, of course, described by the vector function $\mathbf{r} + \delta\mathbf{r}$, where $\delta\mathbf{r}|_{t_1} = \delta\mathbf{r}|_{t_2} = 0$.

If we form the scalar product of the vector $\delta\mathbf{r}$ and the terms of equation (5.99), and integrate from t_1 to t_2, we obtain

$$\int_{t_1}^{t_2} (m\ddot{\mathbf{r}} \cdot \delta\mathbf{r} - \mathbf{F} \cdot \delta\mathbf{r}) dt = 0. \tag{5.100}$$

Applying integration by parts to the first term in equation (5.100), we obtain

$$m\dot{\mathbf{r}} \cdot \delta\mathbf{r}|_{t_1}^{t_2} - m\int_{t_1}^{t_2} \dot{\mathbf{r}} \cdot \delta\dot{\mathbf{r}}dt.$$

The integrated term vanishes because of the properties of $\delta\dot{\mathbf{r}}$. Moreover,

$$\begin{aligned} m\dot{\mathbf{r}} \cdot \delta\mathbf{r} &= \left(\frac{m}{2}\right)(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})dt \\ &= \delta\left(\frac{m}{2}v^2\right) \\ &= \delta T, \end{aligned}$$

where T is the kinetic energy of the moving particle of speed v. Hence equation (5.100) can be rewritten

$$\int_{t_1}^{t_2} (\delta T + \mathbf{F} \cdot \delta\mathbf{r})dt = 0. \tag{5.101}$$

This is **Hamilton's principle** in its general form, as applied to the motion of a single mass particle in a force field that can be either conservative or nonconservative. If $\mathbf{F}$ is conservative, Hamilton's principle assumes an even simpler form. Force $\mathbf{F}$ is conservative, then there exists a scalar function $\phi(x, y, z)$ such that $\mathbf{F} \cdot \delta\mathbf{r} = d\phi$ or equivalently, $\mathbf{F} = \nabla\phi$. The function ϕ is called the **potential function**, and $-\phi$ is (to within an additive constant) the potential energy of the particle in the field. [Note: $\mathbf{F}$ is conservative meaning $\nabla \times \mathbf{F} = 0$ that implies that $\mathbf{F} = \nabla\phi$, because $curlgrade\phi = 0$ automatically and ϕ is called the potential function.] Now,

$$\begin{aligned} \mathbf{F} &= \nabla\phi \\ &= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}. \end{aligned}$$

We know that $\delta\mathbf{r} = \delta x\mathbf{i} + \delta y\mathbf{j} + \delta z\mathbf{k}$, and hence the dot product of two vectors

$$\begin{aligned} \mathbf{F} \cdot \delta\mathbf{r} &= \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z \\ &= \delta\phi, \end{aligned}$$

and therefore, the equation

$$\int_{t_1}^{t_2} (\delta T + \mathbf{F} \cdot \delta\mathbf{r})dt = 0$$

can be rewritten $\int_{t_1}^{t_2} \delta(T + \phi)dt = 0$, or $\delta\int_{t_1}^{t_2} (T + \phi)dt = 0$. And finally, since $\phi = -V$, where V is the potential energy of the system,

$$\delta\int_{t_1}^{t_2} (T - V)dt = 0. \tag{5.102}$$

This is **Hamilton's principle for a single mass particle in a conservative field**. The principle can, of course, be extended to a system of discrete particles by summation and to continuous systems by integration.

Remark

In many elementary problems, a dynamical system is described in terms of coordinates that are distances. Such a choice of coordinates is not always the best one; however, sometimes, problems involving another can be showed more conveniently by choosing other quantities as coordinates, for instance, angles or even areas. In the particular case of a system of P discrete particles, the effect of geometric constraints, assumed to be constant in time, is to reduce the numbers of variables required to specify completely the state of the system at any given instant. To see that a set of constraints actually has such an effect, observe that these rectangular coordinates suffice to determine the position vector (x_j, y_j, z_j) of each mass particle $m_j, j = 1, 2, \ldots p$. If the constraints can be described by $k(<3p)$ constraint and independent equations

$$g_i(x_1, y_1, z_1, \ldots x_p, y_p, z_p) = 0 \quad i = 1, 2, 3, \ldots k \tag{5.103}$$

these k equations may be used, at least theoretically, to eliminate k of the position variables thus leaving only $3p-k$ independent coordinates. As has been mentioned, the variables employed in a particular problem need not be rectangular coordinates. They may be any $3p-k=n$ variables $q_1, q_2, \ldots q_n$ that are independent of one another, which are well suited to a mathematical investigation of the problem at hand, and in terms of which the position of the p particles can be expressed by means of $3p$ equations

$$\left.\begin{aligned} x_j &= x_j(q_1, q_2, \ldots q_n) \\ y_j &= y_j(q_1, q_2, \ldots q_n) \\ z_j &= z_j(q_1, q_2, \ldots q_n) \end{aligned}\right\} \quad j = 1, 2, 3, \ldots p. \tag{5.104}$$

These $3p$ equations in effect ensure that the constraints in equation (5.98) are all compiled with.

Variables $q_1, q_2, \ldots q_n$ of the kind just described are called **generalized coordinates**. A set of generalized coordinates for any particular mechanical system is by no means unique; however, each such set must contain the same number of variables.

(b) Lagrangian equations

Let us investigate the behaviour of a system of p particles a little further. By differentiating equation (5.104) with respect to t, and substituting the results

we obtain

$$\begin{aligned}
\dot{x}_j &= \sum_{i=1}^{n} \frac{\partial x_j}{\partial q_i}\dot{q}_i \\
\dot{y}_j &= \sum_{i=1}^{n} \frac{\partial y_j}{\partial q_i}\dot{q}_i \\
\dot{z}_j &= \sum_{i=1}^{n} \frac{\partial z_j}{\partial q_i}\dot{q}_i
\end{aligned} \tag{5.105}$$

into the expression

$$T = \frac{1}{2}\sum_{j=1}^{p} m_j\left(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2\right), \tag{5.106}$$

we find that the kinetic energy T of the system can be written as a quadratic form

$$T = \dot{q}^T A\dot{q}, \tag{5.107}$$

where $\dot{q}^T = [\dot{q_1}, \dot{q_2}, \ldots \dot{q_n}]$ and the symmetric matrix A is expressed solely in terms of $q_1, q_2, \ldots q_n$. Of course, this implies that T is a homogeneous function of degree 2 in generalized velocity components $\dot{q_1}, \dot{q_2}, \ldots \dot{q_n}$. Now in a conservative system, by definition, the potential energy V depends only on the position of the particles. Hence $V = V(q_1, q_2, \ldots q_n)$ must be a function of the generalized coordinates alone. The function $L = T - V$ is usually referred to as the **Lagrangian** or the kinetic potential. Hamilton's principle, when extended to a conservative system of the particles, may be stated as follows:

During an arbitrary interval $[t_1, t_2]$, the actual motion of a conservative system of particles whose Lagrangian is $T - V = L(q_1, q_2, \ldots q_n, \dot{q_1}, \dot{q_2}, \ldots \dot{q_n})$ is such that **Hamilton's integral**

$$\int_{t_1}^{t_2} (T - V)dt = \int_{t_1}^{t_2} Ldt \tag{5.108}$$

is rendered stationary relative to continuously twice-differentiable functions $q_1, q_2, \ldots q_n$ that take on prescribed values at t_1 and t_2.

From this principle discussed previously, we obtain

$$\left.\begin{aligned}
I &= \int_{t_1}^{t_2} f(t, x_1, x_2, x_n, \ldots, \dot{x_1}, \dot{x_2}, \ldots \dot{x_n})dt \\
&\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x_i}}\right) = 0
\end{aligned}\right\} \quad i = 1, 2, 3, \ldots, n.$$

It follows that if $q_1, q_2, \ldots q_n$ are generalized coordinates of a conservative system of particles they must satisfy the system of Euler–Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_i}}\right) = 0 \quad i = 1, 2, \ldots n. \tag{5.109}$$

These equations are known as **Lagrange's equations of motion or simply as Lagrange's equations**. Since the Lagrangian L of a conservative system does not explicitly involve t, a first integral of system (5.109) is

$$\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = E, \tag{5.110}$$

where E is a constant. Recalling that V is not a function of the generalized velocity components, we have $\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}(T - V) = \frac{\partial T}{\partial \dot{q}_i}$. The series in equation (5.110) thus becomes $\sum_{i=1}^{n} \dot{q}_i \frac{\partial T}{\partial \dot{q}_i}$. Using the fact that T is a homogeneous function of degree 2 in $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n$ and applying Euler's homogeneous function theorem, we find that the sum of the preceding series is $2T$. We may, therefore, write equation (5.110) as

$$2T - (T - V) = T + V = E. \tag{5.111}$$

This result shows that a conservative particle system, when in motion, must move in such a way that the sum of its kinetic and potential energies remains constant. The total energy E of the system is determined when the initial values of all the q_i and $\dot{q}_i$ are assigned. As our earlier expression $\mathbf{F} \cdot \delta \mathbf{r} = \delta \phi$ for the potential energy of a single particle indicates, in a conservative system of discrete particles, in work done by the various forces when the generalized coordinates $\{q_i\}$ of the system under small changes $\{\delta q_i\}$ is

$$\begin{aligned}
\delta \phi &= -\delta V \\
&= -\left[\frac{\partial V}{\partial q_1}\delta q_1 + \frac{\partial V}{\partial q_2}\delta q_2 + \cdots + \frac{\partial V}{\partial q_n}\delta q_n\right] \\
&= Q_1 \delta q_1 + Q_2 \delta q_2 + \cdots + Q_n \delta q_n,
\end{aligned}$$

where we have introduced the conventional symbol Q_i for $-\frac{\partial V}{\partial q_i}$. The typical term in this expression, $Q_i \delta q_i$ is the work done in a displacement in which δq_i is different from zero but all other $\delta q's$ are equal to zero, since q_i is not necessarily a distance, Q_i is not necessarily a force. Nonetheless, the $Q's$ are referred to as **generalized forces**. Using the relation $L = T - V$, $\frac{\partial V}{\partial \dot{q}_i} = 0$, and $\frac{\partial V}{\partial q_i} = -Q_i$, we find that system (5.104) can be written

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right) - \frac{\partial V}{\partial q_i} = -Q_i \quad i = 1, 2, \ldots, n. \tag{5.112}$$

In a nonconservative system, V as well as T may involve the generalized coordinates, in which case, the relation $\frac{\partial V}{\partial \dot{q}_i} = 0$, and no loss occurs. Nonetheless,

equation (5.112) is still correct, although we shall not prove the fact, the only difference being that in a nonconservative system the generalized forces cannot be derived from a potential function.

5.18 Hamilton's equations

A particle of mass moving freely with velocity $\mathbf{V}$ has momentum $m\mathbf{V}$ and kinetic energy $T = \frac{1}{2}m\mathbf{V}\cdot\mathbf{V}$. In three dimensions, each rectangular component of the momentum equals the derivative of the kinetic energy

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2),$$

with respect to the corresponding velocity components, i.e.

$$\left.\begin{aligned} \frac{\partial T}{\partial \dot{x}} &= m\dot{x} \\ \frac{\partial T}{\partial \dot{y}} &= m\dot{y} \\ \frac{\partial T}{\partial \dot{z}} &= m\dot{z} \end{aligned}\right\}.$$

For a system of particles having generalized coordinates q_i, $1 \le i \le n$, and kinetic energy

$$\begin{gathered} T(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n), \text{ the variables} \\ \frac{\partial T}{\partial \dot{q}_i} = p_i \quad 1 \le i \le n, \end{gathered} \tag{5.113}$$

are by analogy, referred to as generalized moments or generalized momentum coordinates, although those quantities may or may not have the dimensions of momentum. The kinetic energy T of the system is a quadratic form

$$T = \dot{q}^T A \dot{q} \quad A = A^T, \tag{5.114}$$

in the generalized velocity components $\dot{q}_i$ thus, Euler's homogeneous function theorem, in conjunction with equation (5.109), yields

$$\sum_{i=1}^{n} \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = \sum_{i=1}^{n} p_i \dot{q}_i = 2T, \tag{5.115}$$

which shows that each product $p_i\dot{q}_i$ has the dimensions of energy regardless of the nature of the generalized coordinates.

$$T = \begin{bmatrix} \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n \end{bmatrix} \begin{bmatrix} a_{11} & \ldots & 0 \\ 0 & a_{22} & \ldots 0 \\ \ldots & .. & a_{nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$= \begin{bmatrix} \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n \end{bmatrix} \begin{bmatrix} a_{11} & \dot{q}_1 \\ a_{22} & \dot{q}_2 \\ \vdots & \vdots \\ a_{nn} & \dot{q}_n \end{bmatrix},$$

and thus we obtain

$$T = \dot{q}_1 a_{11} \dot{q}_1 + \dot{q}_2 a_{22} \dot{q}_2 + \cdots + \dot{q}_n a_{nn} \dot{q}_n]$$

$$\dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2\dot{q}_i a_{ii} \dot{q}_i$$

$$\sum_{i=1}^{n} \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2 \sum \dot{q}_i a_{ii} \dot{q}_i = 2T.$$

Differentiating equation (5.115) with respect to $\dot{q}_i$ and noting that $a_{ij} = a_{ji}$, we get

$$\begin{aligned}
\frac{\partial T}{\partial \dot{q}_i} &= p_i \\
&= \frac{\partial}{\partial \dot{q}_i}(\dot{q}^T) A \dot{q} + \dot{q}^T A \frac{\partial}{\partial \dot{q}_i}(\dot{q}) \\
&= \begin{bmatrix} a_{11} \; a_{12} \; \ldots \; a_{1n} \end{bmatrix} \dot{q} + \dot{q}^T \begin{bmatrix} a_{11} \; a_{12} \; \ldots \; a_{1n} \end{bmatrix}^T \\
&= 2 \begin{bmatrix} a_{11}\dot{q}_1 \; a_{12}\dot{q}_2 \; \ldots \; a_{1n}\dot{q}_n \end{bmatrix} \quad 1 \le i \le n. \qquad (5.116)
\end{aligned}$$

In matrix form, these n equations read

$$p = 2A\dot{q}. \qquad (5.117)$$

Both equations (5.116) and (5.117) determine each p_i as a linear homogeneous function of $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$. Solving equation (5.117) for $\dot{q}$,

$$\dot{q} = \tfrac{1}{2} A^{-1} p. \qquad (5.118)$$

Let us denote these functions by

$$\dot{q}_i = f_i(q_1, q_2, q_3, \ldots, q_n, p_1, p_2, p_3, \ldots, p_n). \tag{5.119}$$

For a conservative system of particles having total energy E, potential energy V, and Lagrangian L,

$$E = T + V = 2T - (T - V) = 2T - L, \tag{5.120}$$

or substituting from equation (5.115) for $2T$,

$$E = \sum_{i=1}^{n} p_i \dot{q}_i - L. \tag{5.121}$$

When each $\dot{q}_i$ in the right-hand member of this equation is replaced by the corresponding function f_i of equation (5.119), the total energy E is transformed into a function H of the variables $q_1, q_2, q_3, \ldots, q_n, p_1, p_2, p_3, \ldots, p_n$ called the Hamiltonian particle system. Thus,

$$\begin{aligned} &H(q_1, q_2, q_3, \ldots, q_n, p_1, p_2, p_3, \ldots, p_n) \\ &\quad = \sum_{i=1}^{n} p_i \dot{q}_i - L(q_1, q_2, q_3, \ldots, q_n, \dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_n), \end{aligned} \tag{5.122}$$

where in the right-hand member of the equation, each q_i stands for the corresponding function $f_i(q_1, q_2, q_3, \ \ldots, q_n, p_1, p_2, p_3, \ \ldots, p_n)$ of equation (5.119). Moreover, from the first of equation (5.120)

$$\begin{aligned} &H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) \\ &\quad = T(q_1, q_2, \ldots, q_n, f_1, f_2, \ldots, f_n) + V(q_1, q_2, \ldots, q_n), \end{aligned} \tag{5.123}$$

that is to say, the Hamiltonian of a conservative system is the sum of the kinetic and potential energies when the kinetic energy is expressed in terms of q_i and p_i instead of the q_i and $\dot{q}_i$. Differentiating the Hamiltonian equation (5.117) partially with respect to p_i, we have

$$\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}_i + \textstyle\sum_{j=1}^{n} p_j \dfrac{\partial \dot{q}_j}{\partial p_i} - \textstyle\sum_{j=1}^{n} \dfrac{\partial L}{\partial \dot{q}_j} \dfrac{\partial \dot{q}_j}{\partial p_i} \\ &= \dot{q}_i + \textstyle\sum_{j=1}^{n} \left[p_j - \dfrac{\partial L}{\partial \dot{q}_j} \right] \dfrac{\partial \dot{q}_j}{\partial p_i}. \end{aligned} \tag{5.124}$$

Since no $\dot{q}_j$ in an argument of V,

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = p_j, \quad \text{i.e.}\ p_j - \frac{\partial L}{\partial \dot{q}_j} = 0 \quad 1 \le j \le n.$$

Hence, equation (5.124) reduces to

$$\frac{\partial H}{\partial p_i} = q_i \quad 1 \leq i \leq n. \tag{5.125}$$

The n equations show that the partial derivatives of the Hamiltonian with respect to p_i in the *ith* component f_i of vector function $\dot{q}$ set from equation (5.118) as a solution of equation (5.116) or equation (5.117). However, until the Hamiltonian is known, the components of $\dot{q}$, as given by equation (5.125), remain indeterminate. The most convenient way of forming the Hamiltonian of a given system is

- Express the potential energy V in terms of q_i and the kinetic energy T in terms of q_i and $\dot{q}_i$.
- Form and solve the n equations, i.e. equation (5.117) of the $\dot{q}_i$ in terms of the q_i and p_i.
- Substitute for the $\dot{q}_i$ in T to obtain

$$H = T + V \quad \text{in terms of} \quad q_1, q_2 \ldots q_n,\ p_1, p_2 \ldots p_n.$$

Using equation (5.122) to express the Lagrangian in terms of the Hamiltonian, we get

$$L = \sum_{i=1}^{n} p_i \dot{q}_i - H,$$

the related Hamilton integral is

$$\int_{t_1}^{t_2} \left\{ \sum_{i=1}^{n} p_i \dot{q}_i - H \right\} dt. \tag{5.126}$$

According to Hamilton's principle, this integral is rendered stationary by the $2n$ continuously differentiable functions $q_1, q_2 \ldots q_n,\ p_1, p_2 \ldots p_n$ that characterize the actual motion of the system of particles and satisfy the constraints provided by equation (5.125), namely

$$\dot{q}_i - \frac{\partial H}{\partial p_i} = 0 \quad 1 \leq i \leq n.$$

To derive the differential equations of motions satisfied by the q_i and p_i, we form the modified Lagrangian

$$L^* = \sum_{i=1}^{n} p_i \dot{q}_i - H + \sum_{i=1}^{n} \mu_i(t) \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right),$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are undetermined functions. For $1 \le i \le n$, we have $\partial L^*/\partial \dot{p_i} = 0$. Thus, the two sets of Lagrangian equations

$$\frac{\partial L^*}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{q_i}}\right) = 0 \quad 1 \le i \le n \tag{5.127}$$

$$\frac{\partial L^*}{\partial p_i} - \frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{p_i}}\right) = 0 \quad 1 \le i \le n, \tag{5.128}$$

yield, in turn,

$$-\frac{\partial H}{\partial q_i} - \sum_{j=1}^{n} \mu_j(t) \frac{\partial^2 H}{\partial q_i . \partial p_j} - \frac{d}{dt}\big[p_i + \mu_i(t)\big] = 0 \quad 1 \le i \le n, \tag{5.129}$$

and

$$\dot{q_i} - \frac{\partial H}{\partial p_i} - \sum_{j=1}^{n} \mu_j(t) \frac{\partial^2 H}{\partial p_i . \partial p_j} = 0 \quad 1 \le i \le n. \tag{5.130}$$

Because of the constraint, equation (5.130) reduced to

$$\sum_{j=1}^{n} \frac{\partial^2 H}{\partial p_i . \partial p_j} \mu_j(t) = 0 \quad 1 \le i \le n. \tag{5.131}$$

The coefficients matrix

$$B = \left[\frac{\partial^2 H}{\partial p_i . \partial p_j}\right] \quad 1 \le i \le n \tag{5.132}$$

of this system of linear equations in $\mu_1, \mu_2, \ldots, \mu_n$ is the matrix $\frac{1}{2}A^{-1}$ of equation (5.118) and is therefore nonsingular. Hence, for $1 \le i \le n$, $\mu_i(t) = 0$ and equation (5.129) becomes

$$-\frac{\partial H}{\partial q_i} - p_i = 0.$$

These n equations, together with equation (5.125), form a system of $2n$ first-order differential equations

$$\dot{q_i} = \frac{\partial H}{\partial p_i} \quad 1 \le i \le n \tag{5.133}$$

$$\dot{p_i} = -\frac{\partial H}{\partial q_i} \quad 1 \le i \le n, \tag{5.134}$$

known as **Hamilton's equations**, or the canonical form of the equations of motion. As we noted earlier, equation (5.133) of Hamilton equations is automatically given by the solution of equation (5.116) for $\dot{q}_i$ in terms of q_i and p_i. Once H is known, these n equations may be checked by computing the partial derivatives of H with respect to the p_i. The other n Hamilton's equations (5.134) require that H be found before their determination. Collectively, Hamilton's equations provide a basis for more sophisticated mathematical techniques applicable to advanced problems of dynamics, celestial mechanics, and atomic structure.

5.19 Some practical problems

In this section, we shall demonstrate the usefulness of the calculus of variations to a number of practical problems in its simplest form.

Example 5.10

Find the shortest distance between two given points in the x–y plane.

Solution

Let us consider the two end-points as $P(x_1, y_1)$ and $Q(x_2, y_2)$. Suppose PRQ is any curve between these two fixed points such that s is the arc length PRQ. Thus, the problem is to determine the curve for which the functional

$$I(y) = \int_P^Q ds \qquad (5.135)$$

is a minimum.

Since $ds/dx = \sqrt{1 + (y')^2}$, then the above integral equation becomes

$$I(y) = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx. \qquad (5.136)$$

In this case, the functional is $f = \sqrt{1 + (y')^2}$, which depends only on y' and so $\partial f/\partial y = 0$. Hence the Euler−Lagrange equation is

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \qquad (5.137)$$

which yields after reduction $y'' = 0$, and the solution of which is the straight line $y = mx + c$. This is a two-parameter family of straight lines. Hence the shortest distance between the two given points is a straight line.

Example 5.11

Determine the Laplace equation from the functional

$$I(u) = \int\int_S \{u_x^2 + u_y^2\} dxdy, \tag{5.138}$$

with a boundary condition $u = f(x, y)$ on S.

Solution

The variational principle gives

$$\delta I = \delta \int\int_S \{u_x^2 + u_y^2\} dxdy = 0. \tag{5.139}$$

This leads to the Euler–Lagrange equation

$$u_{xx} + u_{yy} = 0 \quad in \quad S. \tag{5.140}$$

Similarly, the functional $I\{u(x, y, z)\} = \int\int\int_S (u_x^2 + u_y^2 + u_z^2) dxdydz$ will lead to the three-dimensional Laplace equation.

Example 5.12

Derive the equation of motion for the free vibration of an elastic string of length ℓ.

Solution

The potential energy V of the string is

$$V = \frac{1}{2}\tau \int_0^\ell u_x^2 dx, \tag{5.141}$$

where $u = u(x, t)$ is the displacement of the string from its equilibrium position and τ is the constant tension of the string.

The kinetic energy T is

$$T = \frac{1}{2}\int_0^\ell \rho u_t^2 dx, \tag{5.142}$$

where ρ is the constant line-density of the string.

According to the Hamiltonian principle

$$\begin{aligned}\delta I &= \delta \int_{t_1}^{t_2} (T - V)dt \\ &= \delta \int_{t_1}^{t_2} \int_0^{\ell} \frac{1}{2}(\rho u_t^2 - \tau u_x^2)dxdt \\ &= 0,\end{aligned} \tag{5.143}$$

which has the form $\delta \int_{t_1}^{t_2} \int_0^{\ell} L(u_t, u_x)dxdt$ in which $L = \frac{1}{2}(\rho u_t^2 - \tau u_x^2)$.

The Euler–Lagrange equation is

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) = 0, \tag{5.144}$$

which yields

$$\frac{\partial}{\partial t}(\rho u_t) - \frac{\partial}{\partial x}(\tau u_x) = 0. \tag{5.145}$$

or

$$u_{tt} - c^2 u_{xx} = 0, \tag{5.146}$$

where $c = \sqrt{\tau/\rho}$ is the speed of the string-wave. This is the equation of motion of the elastic string.

Remark

The variational methods can be further extended for a functional depending on functions of more than one independent variable in the form

$$I[u(x,y)] = \int\int_D f(x, y, u, u_x, u_y)dxdy, \tag{5.147}$$

where the values of the function $u(x, y)$ are prescribed on the boundary C of a finite domain D in the x–y plane. We consider the functional f is differentiable and the surface $u = u(x, y)$ yielding an extremum is also twice-differentiable.

The first variation δI of I is defined by

$$\delta I(u, \epsilon) = I(u + \epsilon) - I(u), \tag{5.148}$$

which is by Taylor's expansion theorem

$$\delta I(u, \epsilon) = \int\int_D \{\epsilon f_u + \epsilon_x f_p + \epsilon_y f_q\}dxdy, \tag{5.149}$$

where $\epsilon = \epsilon(x, y)$ is small and $p = u_x$ and $q = u_y$.

According to the variational principle, $\delta I = 0$ for all admissible values of ϵ. The partial integration of the above equation together with $\epsilon = 0$ yields

$$0 = \delta I = \int\int_D \left\{ f_u - \frac{\partial}{\partial x} f_p - \frac{\partial}{\partial y} f_q \right\} \epsilon dxdy. \tag{5.150}$$

This is true for all arbitrary ϵ, and hence the integrand must vanish, that is

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}(f_p) - \frac{\partial}{\partial y}(f_q) = 0. \tag{5.151}$$

This is the Euler–Lagrange equation, that is the second-order PDE to be satisfied by the extremizing function $u(x, y)$.

We have used this information in our previous example.

If the functional $f = f(t, x, y, z, u, u_t, u_x, u_y, u_z)$ where the dependent variable $u = u(t, x, y, z)$ in which $t, x, y,$ and z are the independent variables, then the Euler–Lagrange equation can be written at once as follows:

$$\frac{\partial f}{\partial u} = \frac{\partial}{\partial t}(f_{u_t}) + \frac{\partial}{\partial x}(f_{u_x}) + \frac{\partial}{\partial y}(f_{u_y}) + \frac{\partial}{\partial z}(f_{u_z}). \tag{5.152}$$

Example 5.13

In an optically homogeneous isotropic medium, light travels from one point to another point along a path for which the travel time is minimum. This is known as the **Fermat principle** in optics. Determine its solution.

Solution

The velocity of light v is the same at all points of the medium, and hence the minimum time is equivalent to the minimum path length. For simplicity, consider a path joining the two points A and B in the x–y plane. The time of travel an arc length ds is ds/v. Thus, the variational problem is to find the path for which

$$\begin{aligned} I &= \int_A^B \frac{ds}{v} \\ &= \int_{x_1}^{x_2} \frac{\sqrt{1 + (y')^2}}{v(y)} dx \\ &= \int_{x_1}^{x_2} f(y, y') dx \end{aligned} \tag{5.153}$$

is a minimum, where $y' = \frac{dy}{dx}$, and $v = v(y)$. When f is a function of y and y', then the Euler–Lagrange equation is

$$\frac{d}{dx}(f - y' f_{y'}) = 0. \tag{5.154}$$

This follows from the result

$$
\begin{aligned}
\frac{d}{dx}(f - y'f_{y'}) &= \frac{d}{dx}f(y, y') - y''f_{y'} - y'\frac{d}{dx}(f'_{y}) \\
&= y'f_y + y''f_{y'} - y''f_{y'} - y\frac{d}{dx}(f_{y'}) \\
&= y'\left[f_y - \frac{d}{dx}(f_{y'})\right] \\
&= 0.
\end{aligned}
$$

Hence integrating, we obtain

$$
\begin{aligned}
f - y'f_{y'} &= constant \\
\frac{\sqrt{1 + (y')^2}}{v} - \frac{y'^2}{v\sqrt{1 + (y')^2}} &= constant.
\end{aligned}
$$

After a reduction, the solution becomes $\frac{1}{v\sqrt{1+(y')^2}} = constant$. In order to give the simple physical interpretation, we rewrite this equation in terms of the angle ϕ made by the tangent to the minimum path with vertical y-axis so that $\sin\phi = \frac{1}{\sqrt{1+(y')^2}}$. Hence, $\frac{1}{v}\sin\phi = constant$ for all points on the minimum curve. For a ray of light, $1/v$ must be directly proportional to the refractive index n of the medium through which the light is travelling. This equation is called the *Snell law* of refraction of light. Often, this law can be stated as $n\sin\phi = constant$.

Example 5.14

Derive Newton's second law of motion from Hamilton's principle.

Solution

Let us consider a particle of mass m at a position $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ that is moving under the action of an external force $\mathbf{F}$. The kinetic energy of the particle is $T = \frac{1}{2}m\dot{\mathbf{r}}^2$, and the variation of work done is $\delta W = \mathbf{F}\cdot\delta\mathbf{r}$ and $\delta V = -\delta W$. Then Hamilton's principle for this system is

$$
\begin{aligned}
0 &= \delta\int_{t_1}^{t_2}(T - V)dt \\
&= \int_{t_1}^{t_2}(\delta T - \delta V)dt \\
&= \int_{t_1}^{t_2}(m\dot{\mathbf{r}}\cdot\delta\mathbf{r} + \mathbf{F}\cdot\delta\mathbf{r})dt.
\end{aligned}
$$

Using the integration by parts, this result yields

$$\int_{t_1}^{t_2} (m\ddot{\mathbf{r}} - \mathbf{F}) \cdot \delta\mathbf{r}dt = 0. \tag{5.155}$$

This is true for every virtual displacement $\delta\mathbf{r}$, and hence the integrand must vanish, that is,

$$m\ddot{\mathbf{r}} = \mathbf{F}. \tag{5.156}$$

This is Newton's second law of motion.

5.20 Exercises

1. Solve the following Abel's integral equations:

$$(a)\ \ \pi(x+1) = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt.$$

$$(b)\ \ x + x^3 = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt.$$

$$(c)\ \ \sin x = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt.$$

$$(d)\ \ x^4 = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt.$$

2. Using the method of Laplace transforms or otherwise, solve the following second kind Volterra integral equations:

$$(a)\ \ u(x) = \sqrt{x} - \pi x + 2\int_0^x \frac{1}{\sqrt{x-t}}u(t)dt.$$

$$(b)\ \ u(x) = \frac{1}{2} - \sqrt{x} + \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt.$$

$$(c)\ \ u(x) = 2\sqrt{x} - \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt.$$

$$(d)\ \ u(x) = x + \frac{4}{3}x^{\frac{3}{2}} - \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt.$$

$$(e)\ \ u(x) = 1 + 2\sqrt{x} - \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt.$$

3. Show that the solution of the integral equation

$$\int_0^1 \ln\left|\frac{\sqrt{x}+\sqrt{t}}{\sqrt{x}-\sqrt{t}}\right| u(t)dt = f(x), \quad (0 \le x \le 1)$$

is

$$u(x) = -\frac{1}{\pi^2}\frac{d}{dx}\int_x^1 \frac{d\tau}{\sqrt{(\tau - x)}}\frac{d}{d\tau}\int_0^\tau \frac{f(t)dt}{\sqrt{(\tau - t)}}.$$

4. Show that the solution of the integral equation

$$\int_0^1 \frac{u(t)dt}{|x-t|^\nu} = 1, \quad (0 \le x \le 1)$$

where $0 < \nu < 1$ is

$$u(x) = \frac{1}{\pi}\cos\left(\frac{\pi\nu}{2}\right)\left\{x(1-x)^{(\nu-1)/2}\right\}.$$

5. Show that the solution of the integral equation

$$\int_0^1 \left|x-t\right|^{\frac{1}{2}} u(t)dt = 1, \quad (0 \le x \le 1)$$

is

$$u(x) = x^{-\frac{3}{4}}(1-x)^{-\frac{3}{4}}/(\pi\sqrt{2}), \quad (0 < x < 1.)$$

6. Show that the solution of the following integral equation

$$\mu u(x) = x + \int_0^1 \frac{u(t)dt}{t-x}, \quad (0 < x < 1)$$

where $\mu = -\pi\cot(\pi\nu)$ in which $(0 < \nu < \frac{1}{2})$ is

$$u(x) = \frac{\sin(\pi\nu)}{\pi} x^\nu (1-x)^{-\nu}(\nu - x), \quad (0 < x < 1).$$

7. Show that the solution of the following integral equation

$$\mu u(x) = x + \int_0^1 \frac{u(t)dt}{t-x}, \quad (0 < x < 1)$$

where $\mu = -\pi\cot(\pi\nu)$ in which $(0 < \nu < \frac{1}{2})$ is

$$u(x) = \frac{\sin(\pi\nu)}{\pi} x^{-\nu} (1-x)^{\nu}(\nu + x), \quad (0 < x < 1).$$

8. Show that the minimization of the variational integral

$$\begin{aligned} I &= \int\int_S \left[QT^2 - \alpha\left\{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right\}\right] dS \\ &= \int\int_S f(T, T_x, T_y)dS, \end{aligned}$$

where $T(x,y)$ is the temperature distribution on the surface S, α is the thermal diffusivity, and Q is a constant rate of heat generation, is equivalent to solving the differential equation

$$\frac{\partial}{\partial x}\left(\alpha\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha\frac{\partial T}{\partial y}\right)+QT=0,$$

with the same boundary conditions. [Hint: Use Euler–Lagrange equation to obtain this heat conduction equation.]

9. Show that the maximum and minimum values of the function $f(x,y,z)=xyz$ on the unit sphere $x^2+y^2+z^2=1$ are $\frac{1}{3\sqrt{3}}$ and $-\frac{1}{3\sqrt{3}}$.

10. When a thermonuclear reactor is built in the form of a right circular cylinder, neutron diffusion theory requires its radius and height to satisfy the equation

$$\left(\frac{2.4048}{r}\right)^2+\left(\frac{\pi}{h}\right)^2=k,$$

where k is a constant. Show by Lagrange's principle of extremizing a functional that the values of r and h in terms of k, if the reactor is to occupy as small a volume as possible, are $r=2.4048\sqrt{3/(2k)}$ and $h=\pi\sqrt{3/k}$.

11. Find the maximum value of $f(x,y,z)=\frac{xy+x^2}{z^2+1}$ subject to the constraint $x^2(4-x^2)=y^2$.

12. Derive the Poisson equation $\nabla^2 u=F(x,y)$ from the variational principle with the functional

$$I(u)=\int\int_D\{u_x^2+u_y^2+2uF(x,y)\}dxdy,$$

where $u=u(x,y)$ is given on the boundary C of D.

13. Show that the Euler–Lagrange equation of the variational principle

$$\delta I(u(x,y))=\delta\int\int_D f(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})dxdy=0$$

is

$$f_u-\frac{\partial}{\partial x}(f_{u_x})-\frac{\partial}{\partial y}(f_{u_y})+\frac{\partial^2}{\partial x^2}(f_{u_{xx}})+\frac{\partial^2}{\partial x\partial y}(f_{u_{xy}})+\frac{\partial^2}{\partial y^2}(f_{u_{yy}})=0.$$

14. Derive the Boussinesq equation for water waves

$$u_{tt}-c^2u_{xx}-\mu u_{xxtt}=\tfrac{1}{2}(u^2)_{xx}$$

from the variational principle $\delta\int\int Ldxdt=0$,
where

$$L=\tfrac{1}{2}\phi_t^2-\tfrac{1}{2}c^2\phi_x^2+\tfrac{1}{2}\mu\phi_{xt}^2-\tfrac{1}{6}\phi_x^3$$

and ϕ is the velocity potential defined by $u=\phi_x$.

15. Determine the function representing a curve that makes each of the following equations maximum:

$$(a) \quad I(y(x)) = \int_0^1 (y'^2 + 12xy)dx, \quad y(0) = 0, y(1) = 1.$$

$$(b) \quad I(y(x)) = \int_0^{\pi/2} (y'^2 - y^2)dx, \quad y(0) = 0, y(\pi/2) = 1.$$

16. From the variational principle $\delta \int \int_D L dx dt = 0$ with

$$L = -\rho \int_{-h}^{\eta} \left\{ \phi_t + \frac{1}{2}(\nabla\phi)^2 + gz \right\} dz$$

derive the basic equations of water waves

$$\nabla^2\phi = 0, \quad -h(x,y) < z < \eta(x,y), t > 0$$

$$\eta_t + \nabla\phi \cdot \nabla\eta - \phi_z = 0, \quad on\ z = \eta$$

$$\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gz = 0, \quad on\ z = \eta$$

$$\phi_z = 0, \quad on\ z = -h,$$

where $\phi(x,y,z,t)$ is the velocity potential and $\eta(x,y,t)$ is the free surface displacement function in a fluid of depth h.

17. A mass m under the influence of gravity executes small oscillations about the origin on a frictionless paraboloid $Ax^2 + 2Bxy + Cy^2 = 2z$, where $A > 0$ and $B^2 < AC$, and the positive direction of z-axis is upward. Show that the equation whose solution gives the natural frequencies of the motion is $\omega^4 - g(A + C)\omega^2 + g^2(AC - B^2) = 0$. There are two values of ω^2 unless $A = C$ and $B = 0$. [Hint: Use x and y as the generalized coordinates and take $\dot{z}^2 = 0$.]

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6 Integro-differential equations

6.1 Introduction

This chapter deals with one of the most applied problems in the engineering sciences. It is concerned with the integro-differential equations where both differential and integral operators will appear in the same equation. This type of equations was introduced by Volterra for the first time in the early 1900. Volterra investigated the population growth, focussing his study on the hereditary influences, where through his research work the topic of integro-differential equations was established (see Abramowitz and Stegun [1]).

Scientists and engineers come across the integro-differential equations through their research work in heat and mass diffusion processes, electric circuit problems, neutron diffusion, and biological species coexisting together with increasing and decreasing rates of generating. Applications of the integro-differential equations in electromagnetic theory and dispersive waves and ocean circulations are enormous. More details about the sources where these equations arise can be found in physics, biology, and engineering applications as well as in advanced integral equations literatures. In the electrical *LRC* circuit, one encounters an integro-differential equation to determine the instantaneous current flowing through the circuit with a given voltage $E(t)$, and it is written as $L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\tau)d\tau = E(t)$, with the initial condition $I(0) = I_0$ at $t = 0$. To solve this problem we need to know the appropriate technique (see Rahman [3]).

It s important to note that in the integro-differential equations, the unknown function $u(x)$ and one or more of its derivatives such as $u'(x), u''(x), \ldots$ appear out and under the integral sign as well. One quick source of integro-differential equations can be clearly seen when we convert the differential equation to an integral equation by using Leibnitz rule. The integro-differential equation can be viewed in this case as an intermediate stage when finding an equivalent Volterra integral equation to the given differential equation.

The following are the examples of linear integro-differential equations:

$$u'(x) = f(x) - \int_0^x (x-t)u(t)dt, \quad u(0) = 0 \tag{6.1}$$

$$u''(x) = g(x) + \int_0^x (x-t)u(t)dt, \quad u(0) = 0,\, u'(0) = -1 \tag{6.2}$$

$$u'(x) = e^x - x + \int_0^1 xtu(t)dt, \quad u(0) = 0, \tag{6.3}$$

$$u''(x) = h(x) + \int_0^x tu'(t)dt, \quad u(0) = 0,\, u'(0) = 1. \tag{6.4}$$

It is clear from the above examples that the unknown function $u(x)$ or one of its derivatives appear under the integral sign, and the other derivatives appear out the integral sign as well. These examples can be classified as the Volterra and Fredholm integro-differential equations. Equations (6.1) and (6.2) are the Volterra type whereas equations (6.3) and (6.4) are of Fredholm type integro-differential equations. It is to be noted that these equations are linear integro-differential equations. However, nonlinear integro-differential equations also arise in many scientific and engineering problems. Our concern in this chapter will be linear integro-differential equations and we will be concerned with the different solution techniques. To obtain a solution of the integro-differential equation, we need to specify the initial conditions to determine the unknown constants.

6.2 Volterra integro-differential equations

In this section, we shall present some sophisticated mathematical methods to obtain the solution of the Volterra integro-differential equations. We shall focus our attention to study the integral equation that involves separable kernel of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t) \tag{6.5}$$

We shall first study the case when $K(x,t)$ consists of one product of the functions $g(x)$ and $h(t)$ such that $K(x,t) = g(x)h(t)$ only where other cases can be generalized in the same manner. The nonseparable kernel can be reduced to the separable kernel by using the Taylor's expansion for the kernel involved. We will illustrate the method first and then use the technique to some examples.

6.2.1 The series solution method

Let us consider a standard form of Volterra integro-differential equation of *nth* order as given below:

$$u^{(n)}(x) = f(x) + g(x)\int_0^x h(t)u(t)dt, \quad u^{(n)} = b_k, \quad 0 \le k \le (n-1). \tag{6.6}$$

We shall follow the Frobenius method of series solution used to solve ordinary differential equations around an ordinary point. To achieve this goal, we first assume that the solution $u(x)$ of equation (6.6) is an analytic function and hence can be represented by a series expansion about the ordinary point $x=0$ given by

$$u(x)=\sum_{k=0}^{\infty} a_k x^k, \tag{6.7}$$

where the coefficients a_k are the unknown constants and must be determined. It is to be noted that the first few coefficients a_k can be determined by using the initial conditions so that $a_0=u(0), a_1=u'(0), a_2=\frac{1}{2!}u''(0)$, and so on depending on the number of the initial conditions, whereas the remaining coefficients a_k will be determined from applying the technique as will be discussed later. Substituting equation (6.7) into both sides of equation (6.6) yields

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{(n)} = f(x)+g(x)\int_0^x \left(\sum_{k=0}^{\infty} a_k t^k\right) dt. \tag{6.8}$$

In view of equation (6.8), equation (6.6) will be reduced to calculable integrals in the right-hand side of equation (6.8) that can be easily evaluated where we have to integrate terms of the form $t^n, n\geq 0$ only. The next step is to write the Taylor's expansion for $f(x)$, evaluate the resulting traditional integrals, i.e. equation (6.8), and then equating the coefficients of like powers of x in both sides of the equation. This will lead to a complete determination of the coefficients $a_0, a_1, a_2, \ldots$ of the series in equation (6.7). Consequently, substituting the obtained coefficients $a_k, k\geq 0$ in equation (6.7) produces the solution in the series form. This may give a solution in closed-form, if the expansion obtained is a Taylor's expansion to a well-known elementary function, or we may use the series form solution if a closed form is not attainable.

To give a clear overview of the method just described and how it should be implemented for Volterra integro-differential equations, the series solution method will be illustrated by considering the following example.

Example 6.1

Solve the following Volterra integro-differential equation

$$u''(x)=x\cosh x-\int_0^x tu(t)dt, \quad u(0)=0, \quad u'(0)=1, \tag{6.9}$$

by using the series solution method.

Solution

Substitute $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{6.10}$$

into both sides of the equation (6.9) and using the Taylor's expansion of $\cosh x$, we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = x\left(\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k!)}\right) - \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt. \tag{6.11}$$

Using the initial conditions, we have $a_0 = 0$, and $a_1 = 1$. Evaluating the integrals that involves terms of the form t^n, $n \geq 0$, and using few terms from both sides yield

$$\begin{aligned} 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots = x\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ - \left(\frac{x^3}{3} + \frac{1}{4}a_2 x^4 + \cdots\right). \end{aligned} \tag{6.12}$$

Equating the coefficients of like powers of x in both sides we find $a_2 = 0$, $a_3 = \frac{1}{3!}, a_4 = 0$, and in general $a_{2n} = 0$, for $n \geq 0$ and $a_{2n+1} = \frac{1}{(2n+1)!}$, for $n \geq 0$. Thus, using the values of these coefficients, the solution for $u(x)$ from equation (6.10) can be written in series form as

$$u(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \tag{6.13}$$

and in a closed-form

$$u(x) = \sinh x, \tag{6.14}$$

is the exact solution of equation (6.9).

Example 6.2

Solve the following Volterra integro-differential equation

$$u''(x) = \cosh x + \frac{1}{4} - \frac{1}{4}\cosh 2x + \int_0^x \sinh t u(t) dt, \quad u(0) = 1, u'(0) = 0. \tag{6.15}$$

Solution

Using the same procedure, we obtain the first few terms of the expression $u(x)$ as

$$u(x) = 1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots. \tag{6.16}$$

Substituting equation (6.16) into both sides of equation (6.15) yields

$$\begin{aligned} &2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots \\ &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{1}{4} \\ &- \frac{1}{4}\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots\right) \\ &+ \int_0^x \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)(1 + a_2t^2 + a_3t^3 + \cdots)dt. \end{aligned} \tag{6.17}$$

Integrating the right-hand side and equating the coefficients of like powers of x we find $a_0 = 1, a_1 = 0, a_2 = \frac{1}{2!}, a_3 = 0, a_4 = \frac{1}{4!}, a_5 = 0$, and so on, where the constants a_0 and a_1 are defined by the initial conditions. Consequently, the solution in the series method is given by

$$u(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots, \tag{6.18}$$

which give $u(x) = \cosh x$, as the exact solution in a closed-form.

6.2.2 The decomposition method

In this section, we shall introduce the decomposition method and the modified decomposition method to solve the Volterra integro-differential equations. This method appears to be reliable and effective.

Without loss of generality, we may assume a standard form to Volterra integro-differential equation defined by the standard form

$$u^{(n)} = f(x) + \int_0^x K(x,t)u(t)dt, \quad u^{(k)}(0) = b_k,\ 0 \le k \le (n-1) \tag{6.19}$$

where $u^{(n)}$ is the *nth* order derivative of $u(x)$ with respect to x and b_k are constants that defines the initial conditions. It is natural to seek an expression for $u(x)$ that will be derived from equation (6.19). This can be done by integrating both sides of equation (6.19) from 0 to x as many times as the order of the derivative involved. Consequently, we obtain

$$u(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) + L^{-1}\left(\int_0^x K(x,t)u(t)dt\right), \tag{6.20}$$

where $\sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k$ is obtained by using the initial conditions, and L^{-1} is an n-fold integration operator. Now, we are in a position to apply the decomposition method

by defining the solution $u(x)$ of equation (6.20) on a decomposed series

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{6.21}$$

Substitution of equation (6.21) into both sides of equation (6.20) we get

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) \\ &\quad + L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right) dt\right)\end{aligned} \tag{6.22}$$

This equation can be explicitly written as

$$\begin{aligned}u_0(x) + u_1(x) + u_2(x) + \cdots &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) \\ &\quad + L^{-1}\left(\int_0^x K(x,t)u_0(t)dt\right) \\ &\quad + L^{-1}\left(\int_0^x K(x,t)u_1(t)dt\right) \\ &\quad + L^{-1}\left(\int_0^x K(x,t)u_2(t)dt\right) \\ &\quad + L^{-1}\left(\int_0^x K(x,t)u_3(t)dt\right) \\ &\quad + \cdots\end{aligned} \tag{6.23}$$

The components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ of the unknown function $u(x)$ are determined in a recursive manner, if we set

$$\begin{aligned}u_0(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)), \\ u_1(x) &= L^{-1}\left(\int_0^x K(x,t)u_0(t)dt\right), \\ u_2(x) &= L^{-1}\left(\int_0^x K(x,t)u_1(t)dt\right), \\ u_3(x) &= L^{-1}\left(\int_0^x K(x,t)u_2(t)dt\right), \\ u_4(x) &= L^{-1}\left(\int_0^x K(x,t)u_3(t)dt\right),\end{aligned}$$

and so on. The above equations can be written in a recursive manner as

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) \tag{6.24}$$

$$u_{n+1}(x) = L^{-1}\left(\int_0^x K(x,t)u_n(t)dt\right), \quad n \geq 0 \tag{6.25}$$

In view of equations (6.24) and (6.25), the components $u_0(x), u_1(x), u_2(x), \ldots$ are immediately determined. Once these components are determined, the solution $u(x)$ of equation (6.19) is then obtained as a series form using equation (6.21). The series solution may be put into an exact closed-form solution which can be clarified by some illustration as follows. It is to be noted here that the phenomena of self-cancelling noise terms that was introduced before may be applied here if the noise terms appear in $u_0(x)$ and $u_1(x)$. The following example will explain how we can use the decomposition method.

Example 6.3

Solve the following Volterra integro-differential equation

$$u''(x) = x + \int_0^x (x-t)u(t)dt, \quad u(0) = 0,\, u'(0) = 1, \tag{6.26}$$

by using the decomposition method. Verify the result by the Laplace transform method.

Solution

Applying the two-fold integration operator L^{-1}

$$L^{-1}(.) = \int_0^x \int_0^x (.)dx\, dx, \tag{6.27}$$

to both sides of equation (6.26), i.e. integrating both sides of equation (6.26) twice from 0 to x, and using the given linear conditions yield

$$u(x) = x + \frac{x^3}{3!} + L^{-1}\left(\int_0^x (x-t)u(t)dt\right). \tag{6.28}$$

Following the decomposition scheme, i.e. equations (6.24) and (6.25), we find

$$\begin{aligned}
u_0(x) &= x + \frac{x^3}{3!} \\
u_1(x) &= L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right) \\
&= \frac{x^5}{5!} + \frac{x^7}{7!} \\
u_2(x) &= L^{-1}\left(\int_0^x (x-t)u_1(t)dt\right) \\
&= \frac{x^9}{9!} + \frac{x^{11}}{11!}.
\end{aligned}$$

With this information the final solution can be written as

$$u(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \cdots \tag{6.29}$$

and this leads to $u(x) = \sinh x$, the exact solution in closed-form.

By using the Laplace transform method with the concept of convolution and using the initial conditions the given equation can be very easily simplified to

$$\mathcal{L}\{u(x)\} = \frac{1}{s^2 - 1}$$

and taking the inverse transform, we obtain $u(x) = \sinh x$ which is identical to the previous result.

Example 6.4

Solve the following Volterra integro-differential equation

$$u''(x) = 1 + \int_0^x (x-t)u(t)dt, \quad u(0) = 1, \quad u'(0) = 0, \tag{6.30}$$

by using the decomposition method, then verify it by the Laplace transform method.

Solution

Integrating both sides of equation (6.30) from 0 to x and using the given initial conditions yield

$$u(x) = 1 + \frac{x^2}{2!} + L^{-1}\left(\int_0^x (x-t)u(t)dt\right), \tag{6.31}$$

where L^{-1} is a two-fold integration operator. Following the decomposition method, we obtain

$$
\begin{aligned}
u_0(x) &= 1 + \frac{x^2}{2!} \\
u_1(x) &= L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right) \\
&= \frac{x^4}{4!} + \frac{x^6}{6!} \\
u_2(x) &= L^{-1}\left(\int_0^x (x-t)u_1(t)dt\right) \\
&= \frac{x^8}{8!} + \frac{x^{10}}{10!}
\end{aligned}
$$

Using this information the solution $u(x)$ can be written as

$$
u(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \cdots \tag{6.32}
$$

and this gives $u(x) \cosh x$ the exact solution.

By using the Laplace transform method with the concept of convolution and using the initial conditions, we obtain

$$
\mathcal{L}\{u(x)\} = \frac{s}{s^2 - 1}
$$

and its inversion is simply $u(x) = \cosh x$. These two results are identical.

6.2.3 Converting to Volterra integral equations

This section is concerned with converting to Volterra integral equations. We can easily convert the Volterra integro-differential equation to equivalent Volterra integral equation, provided the kernel is a difference kernel defined by $K(x,t) = K(x-t)$. This can be easily done by integrating both sides of the equation and using the initial conditions. To perform the conversion to a regular Volterra integral equation, we should use the well-known formula described in Chapter 1 that converts multiple integrals into a single integral. We illustrate for the benefit of the reader three

specific formulas:

$$\int_0^x \int_0^x u(t)dt = \int_0^x (x-t)u(t)dt,$$

$$\int_0^x \int_0^x \int_0^x u(t)dt = \frac{1}{2!}\int_0^x (x-t)^2 u(t)dt,$$

$$\underbrace{\int_0^x \int_0^x \cdots \int_0^x u(t)dt}_{n\text{-fold integration}} = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} u(t)dt$$

Having established the transformation to a standard Volterra integral equation, we may proceed using any of the alternative methods discussed before in previous chapters. To give a clear overview of this method we illustrate the following example.

Example 6.5

Solve the following Volterra integro-differential equation

$$u'(x) = 2 - \frac{x^2}{4} + \frac{1}{4}\int_0^x u(t)dt, \quad u(0) = 0. \tag{6.33}$$

by converting to a standard Volterra integral equation.

Solution

Integrating both sides from 0 to x and using the initial condition and also converting the double integral to the single integral, we obtain

$$\begin{aligned} u(x) &= 2x - \frac{x^3}{12} + \frac{1}{4}\int_0^x \int_0^x u(t)dt\,dt \\ &= 2x - \frac{x^3}{12} + \frac{1}{4}\int_0^x (x-t)u(t)dt \end{aligned}$$

It is clearly seen that the above equation is a standard Volterra integral equation. It will be solved by the decomposition method. Following that technique we set

$$u_0(x) = 2x - \frac{x^3}{12}, \tag{6.34}$$

which gives

$$
\begin{aligned}
u_1(x) &= \frac{1}{4}\int_0^x (x-t)\left(2t - \frac{t^3}{12}\right)dt, \\
&= \frac{x^3}{12} - \frac{x^5}{240}.
\end{aligned}
\tag{6.35}
$$

We can easily observed that $\frac{x^3}{12}$ appears with opposite signs in the components $u_0(x)$ and $u_1(x)$, and by cancelling this noise term from $u_0(x)$ and justifying that $u(x) = 2x$, is the exact solution of equation (6.33). This result can be easily verified by taking the Laplace transform of equation (6.33) and using the initial condition which simply reduces to $\mathcal{L}\{u(x)\} = \frac{2}{s^2}$ and its inversion is $u(x) = 2x$.

6.2.4 Converting to initial value problems

In this section, we shall investigate how to reduce the Volterra integro-differential equation to an equivalent initial value problem. In this study, we shall mainly focus our attention to the case where the kernel is a difference kernel of the form $K(x,t) = K(x-t)$. This can be easily achieved by differentiating both sides of the integro-differential equation as many times as needed to remove the integral sign. In differentiating the integral involved we shall use the Leibnitz rule to achieve our goal. The Leibnitz rule has already been introduced in Chapter 1. For ready reference, the rule is

$$
\begin{aligned}
&\text{Let} \quad y(x) = \int_{t=a(x)}^{t=b(x)} f(x,t)dt \\
&\text{then} \quad \frac{dy}{dx} = \int_{t=a(x)}^{t=b(x)} \frac{\partial}{\partial x} f(x,t)dt + \frac{db(x)}{dx} f(b(x),x) - \frac{da(x)}{dx} f(a(x),x).
\end{aligned}
$$

Having converted the Volterra integro-differential equation to an initial value problem, the various methods that are used in any ordinary differential equation can be used to determine the solution. The concept is easy to implement but requires more calculations in comparison to the integral equation technique. To give a clear overview of this method we illustrate the following example.

Example 6.6

Solve the following Volterra integro-differential equation

$$
u'(t) = 1 + \int_0^x u(t)dt, \quad u(0) = 0, \tag{6.36}
$$

by converting it to an initial value problem.

Solution

Differentiating both sides of equation (6.36) with respect to x and using the Leibnitz rule to differentiate the integral at the right-hand side we obtain

$$u''(x) = u(x), \text{with the initial conditions } u(0) = 0, \quad u'(0) = 1, \tag{6.37}$$

where the derivative condition is obtained by substituting $x = 0$ in both sides of the equation (6.36). The solution of equation (6.37) is simply

$$u(x) = A\cosh x + B\sinh x,$$

where A and B are arbitrary constants and using the initial conditions, we have $A = 0$ and $B = 1$ and thus the solution becomes

$$u(x) = \sinh x.$$

This solution can be verified by the Laplace transform method. By taking the Laplace to equation (6.36) and using the initial condition, we have after reduction

$$\mathcal{L}\{u(x)\} = \frac{1}{s^2 - 1}$$

and its inversion gives us $u(x) = \sinh x$ which is identical to the above result.

Example 6.7

Solve the following Volterra integro-differential equation

$$u'(x) = 2 - \frac{x^2}{4} + \frac{1}{4}\int_0^x u(t)dt, \quad u(0) = 0. \tag{6.38}$$

by reducing the equation to an initial value problem.

Solution

By differentiating the above equation with respect to x it can be reduced to the following initial value problem

$$u''(x) - \tfrac{1}{4}u(x) = -\tfrac{x}{2}, \quad u(0) = 0, u'(0) = 2, \tag{6.39}$$

The general solution is obvious and can be written down at once

$$u(x) = A\cosh(x/2) + B\sinh(x/2) + 2x.$$

Using the initial conditions yields $A = B = 0$ and the solution reduces to $u(x) = 2x$. This result can also be obtained by using the Laplace transform method.

6.3 Fredholm integro-differential equations

In this section, we will discuss the reliable methods used to solve Fredholm integro-differential equations. We remark here that we will focus our attention on the equations that involve separable kernels where the kernel $K(x,t)$ can be expressed as the finite sum of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{6.40}$$

Without loss of generality, we will make our analysis on a one-term kernel $K(x,t)$ of the form $K(x,t) = g(x)h(t)$, and this can be generalized for other cases. The non-separable kernel can be reduced to separable kernel by using the Taylor expansion of the kernel involved. We point out that the methods to be discussed are introduced before, but we shall focus on how these methods can be implemented in this type of equations. We shall start with the most practical method.

6.3.1 The direct computation method

Without loss of generality, we assume a standard form to the Fredholm integro-differential equation given by

$$u^{(n)}(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \quad u^{(k)} = b_k(0), 0 \leq k \leq (n-1), \tag{6.41}$$

where $u^{(n)}(x)$ is the *nth* derivative of $u(x)$ with respect to x and b_k are constants that define the initial conditions. Substituting $K(x,t) = g(x)h(t)$ into equation (6.41) yields

$$u^{(n)}(x) = f(x) + g(x)\int_0^1 h(t)u(t)dt, \quad u^{(k)} = b_k, 0 \leq k \leq (n-1). \tag{6.42}$$

We can easily see from equation (6.42) that the definite integral on the right-hand side is a constant α, i.e. we set

$$\alpha = \int_0^1 h(t)u(t)dt, \tag{6.43}$$

and so equation (6.42) can be written as

$$u^{(n)}(x) = f(x) + \alpha g(x). \tag{6.44}$$

It remains to determine the constant α to evaluate the exact solution $u(x)$. To find α, we should derive a form for $u(x)$ by using equation (6.44), followed by substituting the form in equation (6.43). To achieve this we integrate both

sides of equation (6.44) n times from 0 to x, and by using the given initial conditions $u^{(k)} = b_k, 0 \leq k \leq (n-1)$ and we obtain an expression for $u(x)$ in the following form

$$u(x) = p(x; \alpha), \tag{6.45}$$

where $p(x; \alpha)$ is the result derived from integrating equation (6.44) and also by using the given initial conditions. Substituting equation (6.45) into the right-hand side of equation (6.43), integrating and solving the resulting algebraic equation to determine α. The exact solution of equation (6.42) follows immediately upon substituting the value of α into equation (6.45). We consider here to demonstrate the technique with an example.

Example 6.8

Solve the following Fredholm integro-differential equation

$$u'''(x) = \sin x - x - \int_0^{\pi/2} xtu'(t)dt, \tag{6.46}$$

subject to the initial conditions $u(0) = 1$, $u'(0) = 0$, $u''(0) = -1$.

Solution

This equation can be written in the form

$$u'''(x) = \sin x - (1+\alpha)x, \quad u(0) = 1, \quad u'(0) = 0, \quad u''(0) = -1, \tag{6.47}$$

where

$$\alpha = \int_0^{\pi/2} tu'(t)dt. \tag{6.48}$$

To determine α, we should find an expression for $u'(x)$ in terms of x and α to be used in equation (6.47). This can be done by integrating equation (6.47) three times from 0 to x and using the initial conditions; hence, we find

$$u''(x) = -\cos x - \frac{1+\alpha}{2!}x^2$$

$$u'(x) = -\sin x - \frac{1+\alpha}{3!}x^3$$

$$u(x) = \cos x - \frac{1+\alpha}{4!}x^4$$

Substituting the expression for $u'(x)$ into equation (6.48), we obtain

$$\begin{aligned}\alpha &= \int_0^{\pi/2}\left(-t\sin t - \frac{1+\alpha}{3!}t^4\right)dt \\ &= -1.\end{aligned} \tag{6.49}$$

Substituting $\alpha = -1$ into $u(x) = \cos x - \frac{1+\alpha}{4!}x^4$ simply yields $u(x) = \cos x$ which is the required solution of the problem.

6.3.2 The decomposition method

In the previous chapters, the Adomian decomposition method has been extensively introduced for solving Fredholm integral equations. In this section, we shall study how this powerful method can be implemented to determine a series solution to the Fredholm integro-differential equations. We shall assume a standard form to the Fredholm integro-differential equation as given below

$$u^{(n)}(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \quad u^{(k)} = b_k(0),\ 0 \le k \le (n-1), \tag{6.50}$$

Substituting $K(x,t) = g(x)h(t)$ into equation (6.50) yields

$$u^{(n)}(x) = f(x) + g(x)\int_0^1 h(t)u(t)dt. \tag{6.51}$$

Equation (6.51) can be written in the operator form as

$$Lu(x) = f(x) + g(x)\int_0^1 h(t)u(t)dt, \tag{6.52}$$

where the differential operator is given by $L = \frac{d^n}{dx^n}$. It is clear that L is an invertible operator; therefore, the integral operator L^{-1} is an n-fold integration operator and may be considered as definite integrals from 0 to x for each integral. Applying L^{-1} to both sides of equation (6.52) yields

$$\begin{aligned}u(x) &= b_0 + b_1x + \frac{1}{2!}b_2x^2 + \cdots + \frac{1}{(n-1)}b_{n-1}x^{n-1} \\ &\quad + L^{-1}(f(x)) + \left(\int_0^1 h(t)u(t)dt\right)L^{-1}(g(x)).\end{aligned} \tag{6.53}$$

In other words, we integrate equation (6.51) n times from 0 to x and we use the initial conditions at every step of integration. It is important to note that the equation obtained in equation (6.53) is a standard Fredholm integral equation. This information will be used in the later sections.

In the decomposition method, we usually define the solution $u(x)$ of equation (6.50) in a series form given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{6.54}$$

Substituting equation (6.54) into both sides of equation (6.53) we get

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x))\\ &\quad + \left(\int_0^1 h(t)u(t)dt\right) L^{-1}(g(x)),\end{aligned} \tag{6.55}$$

This can be written explicitly as follows:

$$\begin{aligned}u_0(x) + u_1(x) + u_2(x) + \cdots &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x))\\ &\quad + \left(\int_0^1 h(t)u_0(t)dt\right) L^{-1}(g(x))\\ &\quad + \left(\int_0^1 h(t)u_1(t)dt\right) L^{-1}(g(x))\\ &\quad + \left(\int_0^1 h(t)u_2(t)dt\right) L^{-1}(g(x))\\ &\quad + \cdots .\end{aligned} \tag{6.56}$$

The components $u_0(x), u_1(x), u_2(x), \ldots$ of the unknown function $u(x)$ are determined in a recurrent manner, in a similar fashion as discussed before, if we set

$$\begin{aligned}u_0(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x))\\ u_1(x) &= \left(\int_0^1 h(t)u_0(t)dt\right) L^{-1}(g(x))\\ u_2(x) &= \left(\int_0^1 h(t)u_1(t)dt\right) L^{-1}(g(x))\\ u_3(x) &= \left(\int_0^1 h(t)u_2(t)dt\right) L^{-1}(g(x))\\ \cdots\cdots &= \cdots\cdots .\end{aligned} \tag{6.57}$$

The above scheme can be written in compact form as follows:

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x))$$

$$u_{n+1}(x) = \left(\int_0^1 h(t) u_n(t) dt \right) L^{-1}(g(x)), \quad n \geq 0. \tag{6.58}$$

In view of equation (6.58), the components of $u(x)$ are immediately determined and consequently the solution $u(x)$ is determined. The series solution is proven to be convergent. Sometimes the series gives an exact expression for $u(x)$. The decomposition method avoids massive computational work and difficulties that arise from other methods. The computational work can be minimized, sometimes, by observing the so-called self-cancelling noise terms phenomena.

Remark

The noise terms phenomena
The phenomena of the self-cancelling noise terms was introduced by Adomian and Rach [2] and it was proved that the exact solution of any integral or integro-differential equation, for some cases, may be obtained by considering the first two components u_0 and u_1 only. Instead of evaluating several components, it is useful to examine the first two components. If we observe the appearance of like terms in both the components with opposite signs, then by cancelling these terms, the remaining noncancelled terms of u_0 may in some cases provide the exact solution. This can be justified through substitution. The self-cancelling terms between the components u_0 and u_1 are called the noise terms. However, if the exact solution is not attainable by using this phenomena, then we should continue determining other components of $u(x)$ to get a closed-form solution or an approximate solution. We shall now consider to demonstrate this method by an example.

Example 6.9

Solve the following Fredholm integro-differential equation

$$u'''(x) = \sin x - x - \int_0^{\pi/2} xtu'(t)dt, \quad u(0) = 1,\ u'(0) = 0,\ u''(0) = -1, \tag{6.59}$$

by using the decomposition method.

Solution

Integrating both sides of equation (6.59) from 0 to x three times and using the initial conditions we obtain

$$u(x) = \cos x - \frac{x^4}{4!} - \frac{x^4}{4!} \int_0^{\pi/2} tu'(t)dt. \tag{6.60}$$

We use the series solution given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{6.61}$$

Substituting equation (6.61) into both sides of equation (6.60) yields

$$\sum_{n=0}^{\infty} u_n(x) = \cos x - \frac{x^4}{4!} - \frac{x^4}{4!}\int_0^{\pi/2} t\left(\sum_{n=0}^{\infty} u'_n(t)\right) dt. \tag{6.62}$$

This can be explicitly written as

$$\begin{aligned} u_0(x) + u_1(x) + u_2(x) + \cdots = \cos x - \frac{x^4}{4!} - \frac{x^4}{4!}\left(\int_0^{\pi/2} tu'_0(t)dt\right) \\ - \frac{x^4}{4!}\left(\int_0^{\pi/2} tu'_1(t)dt\right) - \frac{x^4}{4!}\left(\int_0^{\pi/2} tu'_2(t)dt\right) \\ + \cdots . \end{aligned} \tag{6.63}$$

Let us set

$$u_0(x) = \cos x - \frac{x^4}{4!}, \tag{6.64}$$

$$\begin{aligned} u_1(x) &= -\frac{x^4}{4!}\int_0^{\pi/2} t\left(-\sin t - \frac{t^3}{3!}\right) dt \\ &= \frac{x^4}{4!} + \frac{\pi^5}{(5!)(3!)(32)}x^4. \end{aligned} \tag{6.65}$$

Considering the first two components $u_0(x)$ and $u_1(x)$ in equations (6.64) and (6.65), we observe that the term $\frac{x^4}{4!}$ appears in both components with opposite signs. Thus, according to the noise phenomena the exact solution is $u(x) = \cos x$. And this can be easily verified to be true.

6.3.3 Converting to Fredholm integral equations

This section is concerned about a technique that will reduce Fredholm integro-differential equation to an equivalent Fredholm integral equation. This can be easily done by integrating both sides of the integro-differential equation as many times as the order of the derivative involved in the equation from 0 to x for every time we

integrate, and using the given initial conditions. It is worth noting that this method is applicable only if the Fredholm integro-differential equation involves the unknown function $u(x)$ only, and not any of its derivatives, under the integral sign.

Having established the transformation to a standard Fredholm integral equation, we may proceed using any of the alternative method, namely the decomposition method, direct composition method, the successive approximation method, or the method of successive substitutions. We illustrate an example below.

Example 6.10

Solve the following Fredholm integro-differential equation

$$u''(x) = e^x - x + x\int_0^1 tu(t)dt, \quad u(0) = 1, \quad u'(0) = 1, \tag{6.66}$$

by reducing it to a Fredholm integral equation.

Solution

Integrating both sides of equation (6.66) twice from 0 to x and using the initial conditions we obtain

$$u(x) = e^x - \frac{x^3}{3!} + \frac{x^3}{3!}\int_0^1 tu(t)dt, \tag{6.67}$$

a typical Fredholm integral equation. By the direct computational method, this equation can be written as

$$u(x) = e^x - \frac{x^3}{3!} + \alpha\frac{x^3}{3!}, \tag{6.68}$$

where the constant α is determined by

$$\alpha = \int_0^1 tu(t)dt, \tag{6.69}$$

Substituting equation (6.68) into equation (6.69) we obtain

$$\alpha = \int_0^1 t\left(e^t - \frac{t^3}{3!} + \alpha\frac{t^3}{3!}\right)dt,$$

which reduces to yield $\alpha = 1$. Thus, the solution can be written as $u(x) = e^x$.

Remark

It is worth noting that the main ideas we applied are the direct computation method and the decomposition method, where the noise term phenomena was introduced.

The direct computation method provides the solution in a closed-form, but the decomposition method provides the solution in a rapidly convergent series.

6.4 The Laplace transform method

To solve the linear integro-differential equation with initial conditions, the method of Laplace transform plays a very important role in many engineering problems specially in electrical engineering circuit. We shall discuss a fundamental first order integro-differential equation arising in the *LRC* circuit theory. We shall demonstrate its solution taking a very general circuit problem.

The current flowing through a *LRC* electric circuit with voltage $E(t)$ is given by the following integro-differential equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\tau)d\tau = E(t), \quad I(0) = I_0 \tag{6.70}$$

where $I(t)$ is the current, $E(t)$ the voltage, L the inductance, R the resistance, and C the capacitance of the circuit. Here, L, R, C are all constants. And $I(0) = I_0$ is the initial condition with I_0 a constant current.

Now to determine the instantaneous current flowing through this circuit, we shall use the method of Laplace transform in conjunction with the convolution integral. We define the Laplace transform as $\mathcal{L}\{I(t)\} = \int_0^\infty e^{-st}I(t)dt$. Thus, taking the Laplace transform to both sides of the equation (6.70) and using the Laplace transform property that $\mathcal{L}\{\frac{dI}{dt}\} = s\mathcal{L}\{I(t)\} - I_0$, we obtain

$$Ls\mathcal{L}\{I(t)\} - LI_0 + R\mathcal{L}\{I(t)\} + \tfrac{1}{Cs}\mathcal{L}\{I(t)\} = \mathcal{L}\{E(t)\}. \tag{6.71}$$

Solving for $\mathcal{L}\{I(t)\}$, we obtain

$$\mathcal{L}\{I(t)\} = \frac{s}{Ls^2 + Rs + \frac{1}{C}}\{E(t)\} + (LI_0)\frac{s}{Ls^2 + Rs + \frac{1}{C}} \tag{6.72}$$

To invert the Laplace transform of $\frac{s}{Ls^2+Rs+\frac{1}{C}}$ we need special attention. In the following we show the steps to be considered in this inverse process.

$$\begin{aligned}\frac{s}{Ls^2 + Rs + \frac{1}{C}} &= \frac{1}{L}\left\{\frac{s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right\}\\ &= \frac{1}{L}\left\{\frac{s}{(s + \frac{R}{2L})^2 + (\frac{1}{LC} - \frac{R^2}{4L^2})}\right\}\end{aligned}$$

To find the Laplace inverse of the above transform we need to consider three important cases.

$$\text{(a)}\ \ R = 2\sqrt{\frac{L}{C}}$$

$$\text{(b)}\ \ R < 2\sqrt{\frac{L}{C}}$$

$$\text{(c)}\ \ R > 2\sqrt{\frac{L}{C}}.$$

The Laplace inverse in case (a) $R = 2\sqrt{\frac{L}{C}}$ is obtained as

$$\begin{aligned}\frac{1}{L}\mathcal{L}^{-1}\frac{s}{(s+\frac{R}{2L})^2} &= \frac{1}{L}e^{-\frac{Rt}{2L}}\mathcal{L}^{-1}\left(\frac{s-\frac{R}{2L}}{s^2}\right)\\ &= \frac{1}{L}e^{-\frac{Rt}{2L}}\left(1-\frac{Rt}{2L}\right)\end{aligned} \tag{6.73}$$

The Laplace inverse in case (b) $R < 2\sqrt{\frac{L}{C}}$ is the following:

Let $\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$, then we obtain

$$\begin{aligned}\frac{1}{L}\mathcal{L}^{-1}\left\{\frac{s}{(s+\frac{R}{2L})^2+\omega^2}\right\} &= \frac{1}{L}e^{-\frac{Rt}{2L}}\mathcal{L}^{-1}\left(\frac{s-\frac{R}{2L}}{s^2+\omega^2}\right)\\ &= \frac{1}{L}e^{-\frac{Rt}{2L}}\left(\cos\omega t - \frac{R}{2L\omega}\sin\omega t\right)\end{aligned}$$

The Laplace inverse in case (c) $R > 2\sqrt{\frac{L}{C}}$ is the following:

Let $\lambda = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$, then we obtain

$$\begin{aligned}\frac{1}{L}\mathcal{L}^{-1}\left\{\frac{s}{(s+\frac{R}{2L})^2-\lambda^2}\right\} &= \frac{1}{L}e^{-\frac{Rt}{2L}}\mathcal{L}^{-1}\left(\frac{s-\frac{R}{2L}}{s^2-\lambda^2}\right)\\ &= \frac{1}{L}e^{-\frac{Rt}{2L}}\left(\cosh\lambda t - \frac{R}{2L\lambda}\sinh\lambda t\right).\end{aligned}$$

Hence the solution to the linear integro-differential equation for the circuit problem can be written immediately as follows:

The solution in case (a) is

$$I(t) = \left(\frac{1}{L}\right)\int_0^t e^{-\frac{R\tau}{2L}}\left(1-\frac{R\tau}{2L}\right)E(t-\tau)d\tau + I_0e^{-\frac{Rt}{2L}}\left(1-\frac{Rt}{2L}\right) \tag{6.74}$$

The solution in case (b) is

$$I(t) = \left(\frac{1}{L}\right) \int_0^t e^{-\frac{R\tau}{2L}} \left(\cos \omega\tau - \frac{R}{2L\omega} \sin \omega\tau\right) E(t-\tau) d\tau$$

$$+ I_0 e^{-\frac{Rt}{2L}} \left(\cos \omega t - \frac{R}{2L\omega} \sin \omega t\right) \tag{6.75}$$

The solution in case (c) is

$$I(t) = \left(\frac{1}{L}\right) \int_0^t e^{-\frac{R\tau}{2L}} \left(\cosh \lambda\tau - \frac{R}{2L\lambda} \sinh \lambda\tau\right) E(t-\tau) d\tau$$

$$+ I_0 e^{-\frac{Rt}{2L}} \left(\cosh \lambda t - \frac{R}{2L\omega} \sinh \lambda t\right) \tag{6.76}$$

Some special cases

If $E(t) = E_0$ a constant, then the solutions for the cases (a), (b), and (c) are given by

$$(a)\;\; I(t) = \left(\frac{E_0 t}{L} + I_0\left(1 - \frac{Rt}{2L}\right)\right) e^{-\frac{Rt}{2L}}$$

$$(b)\;\; I(t) = \left(\frac{E_0}{L\omega}\right) e^{-\frac{Rt}{2L}} \sin \omega t + I_0 e^{-\frac{Rt}{2L}} \left(\cos \omega t - \frac{R}{2L\omega} \sin \omega t\right)$$

$$(c)\;\; I(t) = \left(\frac{E_0}{L\lambda}\right) e^{-\frac{Rt}{2L}} \sinh \lambda t + I_0 e^{-\frac{Rt}{2L}} \left(\cosh \omega t - \frac{R}{2L\lambda} \sinh \lambda t\right)$$

If $E(t) = E_0\delta(t)$, then the solutions for the cases (a), (b), and (c) are, respectively, given by

$$\text{(a)}\;\; I(t) = \left(\frac{E_0}{L} + I_0\right) e^{-\frac{Rt}{2L}} \left(1 - \frac{Rt}{2L}\right)$$

$$\text{(b)}\;\; I(t) = \left(\frac{E_0}{L} + I_0\right) e^{-\frac{Rt}{2L}} \left(\cos \omega t - \frac{R}{2L\omega} \sin \omega t\right)$$

$$\text{(c)}\;\; I(t) = \left(\frac{E_0}{L} + I_0\right) e^{-\frac{Rt}{2L}} \left(\cosh \lambda t - \frac{R}{2L\lambda} \sinh \lambda t\right)$$

These solutions are, respectively, called (a) the **critically damped**, (b) the **damped oscillatory**, and (c) the **over damped**.

6.5 Exercises

Solve the following Fredholm integro-differential equations by using the *direct computation method***:**

1. $u'(x) = \frac{x}{2} - \int_0^1 xtu(t)dt, \quad u(0) = \frac{1}{6}.$
2. $u''(x) = -\sin x + x - \int_0^{\pi/2} xtu(t)dt, \quad u(0) = 0, u'(0) = 1.$
3. $u'(x) = 2\sec^2 x \tan x - x + \int_0^{\pi/4} xtu(t)dt, \quad u(0) = 1.$

Solve the following Fredholm integro-differential equations by using the *decomposition method***:**

4. $u'(x) = xe^x + e^x - x + \int_0^1 xu(t)dt, \quad u(0) = 0.$
5. $u''(x) = -\sin x + x - \int_0^{\pi/2} xtu(t)dt, \quad u(0) = 0, u'(0) = 1.$
6. $u'''(x) = -\cos x + x + \int_0^{\pi/2} xu''(t)dt, \quad u(0) = 0, u'(0) = 1, u''(0) = 0.$

Solve the following integro-differential equations:

7. $u'(x) = 1 - 2x\sin x + \int_0^x u(t)dt, \quad u(0) = 0.$
8. $u''(x) = 1 - x(\cos x + \sin x) - \int_0^x tu(t)dt, \quad u(0) = -1, u'(0) = 1.$
9. $u''(x) = \frac{x^2}{2} - x\cosh x - \int_0^x tu(t)dt, \quad u(0) = 1, u'(0) = -1.$

Solve the following Volterra integro-differential equations by converting problem to an initial value problem:

10. $u'(x) = e^x - \int_0^x u(t)dt, \quad u(0) = 1.$
11. $u''(x) = -x - \frac{x^2}{2!} + \int_0^x (x-t)u(t)dt, \quad u(0) = 1, u'(0) = 1.$
12. $u'(x) = 1 + \sin x + \int_0^x u(t)dt, \quad u(0) = -1.$

References

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7 Symmetric kernels and orthogonal systems of functions

7.1 Development of Green's function in one-dimension

Green's functions serve as mathematical characterizations of important physical concepts. These functions are of fundamental importance in many practical problems, and in understanding of the theory of differential equations. In the following we illustrate a simple practical problem from mechanics whose solution can be composed of a Green's function, named for the English mathematical physicist George Green (1793–1841).

7.1.1 A distributed load of the string

We consider a perfectly flexible elastic string stretched to a length l under tension T. Let the string bear a distributed load per unit length $\omega(x)$ including the weight of the string. Also assume that the static deflections produced by this load are all perpendicular to the original, undeflected position of the strings.

Consider an elementary length Δx of the string as shown in Figure 7.1. Since the deflected string is in equilibrium, the net horizontal and vertical forces must both be zero. Thus,

$$F_1 \cos\alpha_1 = F_2 \cos\alpha_2 = T \tag{7.1}$$

$$F_2 \sin\alpha_2 = F_1 \sin\alpha_1 - \omega(x)\Delta x \tag{7.2}$$

Here, the positive direction of $\omega(x)$ has been chosen upward. We consider the deflection to be so small that the forms F_1, F_2 will not differ much from the tension of string T, i.e. $T = F_1 = F_2$. So, we obtain approximately

$$T \sin\alpha_2 = T \sin\alpha_1 - \omega(x)\Delta x$$

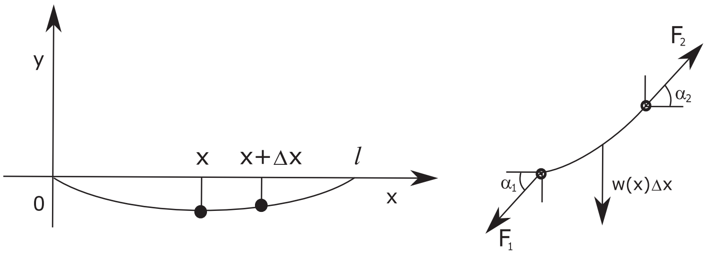

Figure 7.1: A stretched string deflected by a distributed load.

which is

$$\tan \alpha_2 = \tan \alpha_1 - \frac{\omega(x)\Delta x}{T} \tag{7.3}$$

(Using $\sin \alpha \simeq \tan \alpha$ if α is small.) But we know

$$\tan \alpha_2 = \frac{dy}{dx}|_{x+\Delta x}$$
$$\tan \alpha_1 = \frac{dy}{dx}|_{x}$$

Hence rewriting equation (7.3), and letting $\Delta x \to 0$, we obtain

$$\lim_{\Delta x \to 0} \frac{\frac{dy}{dx}|_{x+\Delta x} - \frac{dy}{dx}|_{x}}{\Delta x} = -\frac{\omega(x)}{T}$$
$$\text{or,} \quad T\frac{d^2y}{dx^2} = -\omega(x) \tag{7.4}$$

which is the differential equation satisfied by the deflection curve of the string. This equation is obtained using the distributed load of the string.

7.1.2 A concentrated load of the strings

We now consider the deflection of the string under the influence of a concentrated load rather than a distributed load. A concentrated load is, of course, a mathematical fiction which cannot be realized physically. Any nonzero load concentrated at a single point implies an infinite pressure which may break the string. The use of concentrated load in the investigation of physical systems is both common and fruitful.

It can be easily noticed from equation (7.4) that if there is no distributed load ($\omega(x) = 0$) then $y'' = 0$ at all points of the string. This implies that y is a linear function which follows that the deflection curve of the string under the influence of a single concentrated load R consists of two linear segments as shown in Figure 7.2.

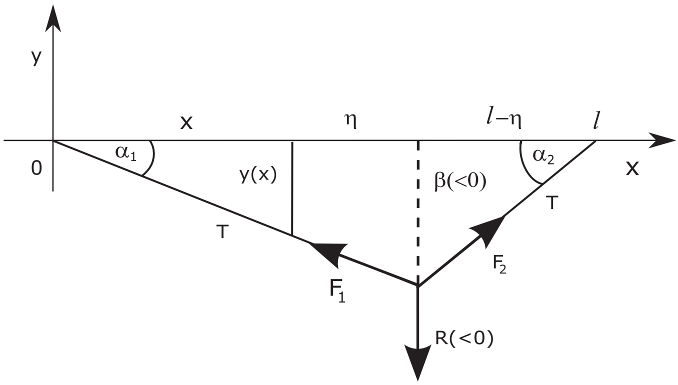

Figure 7.2: A stretched string deflected by a concentrated load.

Resolving the forces acting on the string can be given as

$$\begin{aligned} F_2 \cos\alpha_2 &= F_1 \cos\alpha_1 \\ F_1 \sin\alpha_1 + F_2 \sin\alpha_2 &= -R \end{aligned} \tag{7.5}$$

When the deflection is small we can assume that $F_2 \cos\alpha_2 = F_1 \cos\alpha_1 = T$, or simply $F_1 = F_2 = T$ and

$$\begin{aligned} \sin\alpha_1 &\simeq \tan\alpha_1 \\ \sin\alpha_2 &\simeq \tan\alpha_2 \end{aligned}$$

Then equation (7.5) becomes

$$\begin{aligned} \tan\alpha_1 + \tan\alpha_2 &= -\tfrac{R}{T} \\ \tfrac{-\beta}{\eta} + \tfrac{-\beta}{l-\eta} &= -\tfrac{R}{T} \end{aligned}$$

and

$$\beta = \frac{R(l-\eta)\eta}{Tl}$$

where β is the transverse deflection of the string at $x = \eta$.

With the deflection β known, it is a simple matter to use the similar triangles to find the deflection of the string at any point x. The results are

$$\tfrac{y(x)}{x} = \tfrac{\beta}{\eta} \qquad 0 \le x \le \eta$$

so,

$$y(x) = \frac{R(l-\eta)x}{Tl} \qquad 0 \le x \le \eta$$

also

$$\frac{y(x)}{l-x} = \frac{\beta}{l-\eta}$$

so,

$$y(x) = \frac{R(l-x)\eta}{Tl} \qquad \eta \le x \le l$$

Thus, the results are

$$y(x,\eta) = \begin{cases} \dfrac{R(l-\eta)x}{Tl} & 0 \le x \le \eta \\ \dfrac{R(l-x)\eta}{Tl} & \eta \le x \le l \end{cases} \tag{7.6}$$

It is worth mentioning that $y(x,\eta)$ is used rather than $y(x)$ to indicate that the deflection of y depends on the point η where the concentrated load is applied and the point x where the deflection is observed. These two points are, respectively, called the 'source point' and the 'field point'.

It can be easily observed from equation (7.6) that the deflection of a string at a point x due to a concentrated load R applied at a point η is the same as the deflection produced at the point η by an equal load applied at the point x. When R is a unit load it is customary to use the notation $G(x,\eta)$ known as the Green's function corresponding to the function $y(x,\eta)$. Many authors call Green's function an **influence function**. It is observed that this Green's function is symmetric in the two variables x and η such that

$$G(x,\eta) = G(\eta,x) \tag{7.7}$$

Thus, for this problem, in terms of unit positive load, Green's function is given by

$$G(x,\eta) = \begin{cases} \dfrac{(l-\eta)x}{Tl} & 0 \le x \le \eta \\ \dfrac{(l-x)\eta}{Tl} & \eta \le x \le l \end{cases} \tag{7.8}$$

The symmetry of $G(x,\eta)$ is an important illustration discovered by Maxwell and Rayleigh known as Maxwell–Rayleigh reciprocity law which holds in many physical systems including mechanical and electrical. James Clerk Maxwell (1831–1879) was a Scottish mathematical physicist. Lord Rayleigh (1842–1919) was an English mathematical physicist.

Remark

It is interesting and important to note that with Green's function $G(x,\eta)$ an expression of deflection of a string under an arbitrary distributed load can be found without solving equation (7.4). To see this point clearly, we divide the interval $0 \le x \le l$ into

n subintervals by the points $\eta_0 = 0, \eta_1, \eta_2, \ldots, \eta_n = l$ such that $\Delta\eta_i = \eta_i - \eta_{i-1}$. Let ξ_i be an arbitrary point in the subinterval $\Delta\eta_i$. Let us also consider that the position of the distributed load acting on the subinterval $\Delta\eta_i$, namely $\omega(\xi_i)\Delta\eta_i$, is concentrated at the point $\eta = \xi_i$. The deflection produced at the point x by this load is the product of the load and the deflection produced at x by a unit load at the point $\eta = \xi_i$, which is

$$(\omega(\xi_i)\Delta\eta_i)G(x, \xi_i)$$

Thus, if we add up all the deflections produced at the point x by the various concentrated forces which together approximate the actual distributed load, we obtain the sum

$$\sum_{i=1}^{n} \omega(\xi_i)G(x, \xi_i)\Delta\eta_i$$

This sum becomes an integral when in the limit $\Delta\eta_i \to 0$, and the deflection $y(x)$ at an arbitrary point x is given by

$$y(x) = \int_0^l \omega(\eta)G(x, \eta)d\eta \tag{7.9}$$

Hence, once the function $G(x, \eta)$ is known, the deflection of the string under any piecewise continuous distributed load can be determined at once by the integral equation (7.9). Mathematically, it is clear that equation (7.9) is a solution of the ordinary differential equation

$$Ty'' = -\omega(x)$$

because

$$\begin{aligned} y'' &= \frac{\partial^2}{\partial x^2} \int_0^l \omega(\eta)G(x, \eta)d\eta \\ &= \int_0^l \omega(\eta) \frac{\partial^2 G}{\partial x^2}(x, \eta)d\eta \end{aligned}$$

and so

$$\int_0^l \omega(\eta) \left(T \frac{\partial^2 G}{\partial x^2}(x, \eta) \right) dx = -\omega(x)$$

This is only true provided

$$T \frac{\partial^2 G}{\partial x^2}(x, \eta) = -\delta(x - \eta)$$

such that

$$\int_0^l \omega(\eta)(- \delta(x - \eta))d\eta = -\omega(x)$$

where $\delta(x - \eta)$ is a Dirac delta function.

7.1.3 Properties of Green's function

This generalized function $\delta(x-\eta)$ has the following important properties (see Lighthill's [9] Fourier Analysis and Generalized Function, 1959, and Rahman [12]):

I. $\delta(x-\eta) = \begin{cases} 0 & x \neq \eta \\ \infty & x = \eta \end{cases}$

II. $\int_{-\infty}^{\infty} \delta(x-\eta)dx = 1$

III. If $f(x)$ is a piecewise continuous function in $-\infty < x < \infty$, then

$$\int_{-\infty}^{\infty} f(x)\delta(x-\eta)dx = f(\eta)$$

Thus, integrating $TG_{xx} = -\delta(x-\eta)$ between $x = \eta + 0$ and $x = \eta - 0$ we obtain,

$$T\int_{\eta-0}^{\eta+0} G_{xx}dx = -\int_{\eta-0}^{\eta+0} \delta(x-\eta)dx$$

$$T\left[\left.\frac{\partial G}{\partial x}\right|_{\eta+0} - \left.\frac{\partial G}{\partial x}\right|_{\eta-0}\right] = -1$$

Hence the jump at $x = \eta$ for this problem is

$$\left.\frac{\partial G}{\partial x}\right|_{\eta+0} - \left.\frac{\partial G}{\partial x}\right|_{\eta-0} = -\left(\frac{1}{T}\right),$$

which is the downward jump. Thus, when the tension $T = 1$, this downward jump is -1 at $x = \eta$.

Definition 7.1

A function $f(x)$ has a jump λ at $x = \eta$ if, and only if, the respective right- and left-hand limits $f(\eta+0)$ and $f(\eta-0)$ of $f(x)$ exists as x tends to η, and

$$f(\eta+0) - f(\eta-0) = \lambda$$

This jump λ is said to be upward or downward depending on whether λ is positive or negative. At a point $x = \eta$ when $f(x)$ is continuous, the jump λ is, of course, zero.

With this definition of jump, it is an easy matter to show that $\frac{\partial G}{\partial x}$ has a downward jump of $-\frac{1}{T}$ at $x = \eta$ because we observe from equation (7.8) that

$$\lim_{x\to\eta+0} \frac{\partial G}{\partial x} = \lim_{x\to\eta+0} \frac{-\eta}{lT} = -\frac{\eta}{lT}$$

$$\lim_{x\to\eta-0} \frac{\partial G}{\partial x} = \lim_{x\to\eta-0} \frac{l-\eta}{lT} = \frac{l-\eta}{lT}$$

It is obvious that these limiting values are not equal and their difference is

$$-\frac{\eta}{lT} - \frac{l-\eta}{lT} = -\frac{1}{T}$$

which is a downward jump as asserted. As we have seen $G(x, \eta)$ consists of two linear expressions, it satisfies the linear homogeneous differential equation $Ty'' = 0$ at all points of the interval $0 \le x \le l$ except at $x = \eta$. In fact the second derivative $\frac{\partial^2 G}{\partial x^2}(x, \eta)$ does not exist because $\frac{\partial G}{\partial x}(x, \eta)$ is discontinuous at that point.

The properties of the function $G(x, \eta)$ which we have just observed are not accidental characteristics for just a particular problem. Instead, they are an important class of functions associated with linear differential equations with constants as well as variable coefficients. We define this class of functions with its properties as follows:

Definition 7.2

Consider the second-order homogeneous differential equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{7.10}$$

with the homogeneous boundary conditions

$$\left.\begin{aligned} x = a: &\quad \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ x = b: &\quad \beta_1 y(b) + \beta_2 y'(b) = 0 \end{aligned}\right\} \tag{7.11}$$

where α_1 and α_2 both are not zero, and β_1 and β_2 both are not zero.

Consider a function $G(x, \eta)$ which satisfies the differential equation (7.10) such that

$$G_{xx} + P(x)G_x + Q(x)G = -\delta(x-\eta) \tag{7.12}$$

with the following property

$$x = a: \quad \alpha_1 G(a, \eta) + \alpha_2 G_x(a, \eta) = 0 \tag{7.13}$$

$$x = b: \quad \beta_1 G(b, \eta) + \beta_2 G_x(b, \eta) = 0 \tag{7.14}$$

$$x = \eta \quad G(x, \eta) \text{ is continuous in } a \le x \le b \tag{7.15}$$

and

$$\lim_{x\to\eta+0}\frac{\partial G}{\partial x}-\lim_{x\to\eta-0}\frac{\partial G}{\partial x}=-1 \tag{7.16}$$

has a jump -1 at $x=\eta$.

Then this function $G(x,\eta)$ is called the Green's function of the problem defined by the given differential equation and its boundary conditions.

In the following, we shall demonstrate how to construct a Green's function involving a few given boundary value problems.

Example 7.1

Construct a Green's function for the equation $y''+y=0$ with the boundary conditions $y(0)=y(\frac{\pi}{2})=0$.

Solution

Since $G(x,\eta)$ satisfies the given differential equation such that

$$G_{xx}+G=-\delta(x-\eta)$$

therefore a solution exists in the following form

$$G(x,\eta)=\begin{cases} A\cos x+B\sin x & 0\le x\le\eta \\ C\cos x+D\sin x & \eta\le x\le\frac{\pi}{2}\end{cases}$$

The boundary conditions at $x=0$ and $x=\frac{\pi}{2}$ must be satisfied which yields $G(0,\eta)=A=0$ and $G(\frac{\pi}{2},\eta)=D=0$. Hence we have

$$G(x,\eta)=\begin{cases} B\sin x & 0\le x\le\eta \\ C\cos x & \eta\le x\le\frac{\pi}{2}\end{cases}$$

From the continuity condition at $x=\eta$, we have $B\sin\eta=C\cos\eta$ such that $\frac{B}{\cos\eta}=\frac{C}{\sin\eta}=\alpha$. Thus, $B=\alpha\cos\eta$ and $C=\alpha\sin\eta$ where α is an arbitrary constant. The Green's function reduces to

$$G(x,\eta)=\begin{cases} \alpha\cos\eta\sin x & 0\le x\le\eta \\ \alpha\sin\eta\cos x & \eta\le x\le\frac{\pi}{2}\end{cases}$$

Finally to determine α, we use the jump condition at $x=\eta$ which gives

$$\left.\frac{\partial G}{\partial x}\right|_{\eta+0}-\left.\frac{\partial G}{\partial x}\right|_{\eta-0}=-1$$

$$\alpha[\sin^2\eta+\cos^2\eta]=1$$

and, therefore $\alpha = 1$. With α known, the Green's function is completely determined, and we have

$$G(x,\eta) = \begin{cases} \cos\eta \sin x & 0 \le x \le \eta \\ \cos x \sin\eta & \eta \le x \le \frac{\pi}{2} \end{cases}$$

Example 7.2

Construct the Green's function for the equation $y'' + \nu^2 y = 0$ with the boundary conditions $y(0) = y(l) = 0$.

Solution

Since $G(x,\eta)$ satisfies the given differential equation such that

$$G_{xx} + \nu^2 G = -\delta(x-\eta),$$

therefore a solution can be formed as

$$G(x,\eta) = \begin{cases} A\cos\nu x + B\sin\nu x & 0 \le x \le \eta \\ C\cos\nu x + D\sin\nu x & \eta \le x \le 1 \end{cases}$$

The boundary conditions at $x = 0$ and at $x = 1$ must be satisfied which yields $G(0,\eta) = A = 0$ and $G(1,\eta) = C\cos\nu + D\sin\nu = 0$ such that $\frac{C}{\sin\nu} = \frac{-D}{\cos\nu} = \alpha$ which gives $C = \alpha\sin\nu$ and $D = -\alpha\cos\nu$. Substituting these values, $G(x,\eta)$ is obtained as :

$$G(x,\eta) = \begin{cases} B\sin\nu x & 0 \le x \le \eta \\ \alpha\sin\nu(1-x) & \eta \le x \le 1 \end{cases}$$

where α is an arbitrary constant. To determine the values of B and α we use the continuity condition at $x = \eta$. From the continuity at $x = \eta$, we have

$$B\sin\nu\eta = \alpha\sin\nu(1-\eta)$$

such that

$$\frac{B}{\sin\nu(1-\eta)} = \frac{\alpha}{\sin\nu\eta} = \gamma$$

which yields

$$\alpha = \gamma\sin\nu\eta$$
$$\beta = \gamma\sin\nu(1-\eta)$$

where γ is an arbitrary constant. Thus, the Green's function reduces to

$$G(x,\eta) = \begin{cases} \gamma\sin\nu(1-\eta)\sin\nu x & 0 \le x \le \eta \\ \gamma\sin\nu\eta\sin\nu(1-x) & \eta \le x \le 1 \end{cases}$$

Finally to determine γ, we use the jump condition at $x=\eta$ which is

$$\left.\frac{\partial G}{\partial x}\right|_{\eta+0} - \left.\frac{\partial G}{\partial x}\right|_{\eta-0} = -1$$

From this condition, we can obtain

$$\gamma = \frac{1}{\nu \sin \nu}.$$

With γ known, the Green's function is completely determined, and we have

$$G(x,\eta) = \begin{cases} \dfrac{\sin \nu x \sin \nu(1-\eta)}{\nu \sin \nu} & 0 \le x \le \eta \\ \dfrac{\sin \nu\eta \sin \nu(1-x)}{\nu \sin \nu} & \eta \le x \le 1 \end{cases}$$

This is true provided $\nu \neq n\pi, n=0,1,2,\ldots$. It can be easily seen that $G(x,\eta)=G(\eta,x)$ which is known as the symmetry property of any Green's function. This property is inherent with this function. However, it should be noted that for some physical problems this Green's function may not be symmetric.

Example 7.3

Find the Green's function for the following boundary value problem:

$$y'' + \nu^2 y = 0; \quad y(0) = y(1); \quad y'(0) = y'(1)$$

Solution

As before the Green's function is given by

$$G(x,\eta) = \begin{cases} A \cos \nu x + B \sin \nu x & 0 \le x \le \eta \\ C \cos \nu x + D \sin \nu x & \eta \le x \le 1 \end{cases}$$

The boundary conditions at $x=0$ and $x=1$ must be satisfied which yields, respectively,

$$\begin{aligned} A &= C \cos \nu + D \sin \nu \\ \nu B &= -C\nu \sin \nu + D\nu \cos \nu \quad \nu \neq 0 \end{aligned}$$

Solving for C and D in terms of A and B, we obtain

$$C = \frac{\begin{vmatrix} A & \sin\nu \\ B & \cos\nu \end{vmatrix}}{\begin{vmatrix} \cos\nu & \sin\nu \\ -\sin\nu & \cos\nu \end{vmatrix}} = A\cos\nu - B\sin\nu$$

$$D = \frac{\begin{vmatrix} \cos\nu & A \\ -\sin\nu & B \end{vmatrix}}{\begin{vmatrix} \cos\nu & \sin\nu \\ -\sin\nu & \cos\nu \end{vmatrix}} = A\sin\nu + B\cos\nu$$

After a little reduction, the Green's function can be written as

$$G(x,\eta) = \begin{cases} A\cos\nu x - B\sin\nu x & 0 \le x \le \eta \\ A\cos\nu(1-x) - B\sin\nu(1-x) & \eta \le x \le 1 \end{cases}$$

Now using the continuity condition at $x = \eta$, the constants A and B are determined as

$$A = \alpha(\sin\nu\eta + \sin\nu(1-\eta))$$
$$B = \alpha(-\cos\nu\eta + \cos\nu(1-\eta))$$

where α is an arbitrary constant.

Hence the Green's function is given by

$$G(x,\eta) = \begin{cases} \alpha[(\sin\nu\eta + \sin\nu(1-\eta))\cos\nu x \\ \quad + (\cos\nu(1-\eta) - \cos\nu\eta)\sin\nu x], & 0 \le x \le \eta \\ \alpha[(\sin\nu\eta + \sin\nu(1-\eta))\cos\nu(1-x) \\ \quad - (\cos\nu(1-\eta) - \cos\nu\eta)\sin\nu(1-x)], & \eta \le x \le 1 \end{cases}$$

Now to determine the value of α, we use the jump condition at $x = \eta$

$$\left.\frac{\partial G}{\partial x}\right|_{\eta+0} - \left.\frac{\partial G}{\partial x}\right|_{\eta-0} = -1$$

And after considerable reduction, we obtain $2\alpha\nu(1 - \cos\nu) = -1$ such that $\alpha = \frac{-1}{2\nu(1-\cos\nu)}$

With α known, the Green's function is completely determined, and we have

$$G(x,\eta) = \begin{cases} \alpha\{\sin\nu(\eta - x) + \sin\nu(1-\eta+x)\} & 0 \le x \le \eta \\ \alpha\{\sin\nu(x-\eta) + \sin\nu(1+\eta-x)\} & \eta \le x \le 1 \end{cases}$$

where $\alpha = \frac{-1}{2\nu(1-\cos\nu)}$.

7.2 Green's function using the variation of parameters

In this section, we shall explore the results of the method of variation of parameters to obtain the Green's function. Let us consider the linear nonhomogeneous second-order differential equation:

$$y'' + P(x)y' + Q(x)y = f(x) \quad a \le x \le b \tag{7.17}$$

with the homogeneous boundary conditions

$$x = a: \quad \alpha_1 y(a) + \alpha_2 y'(a) = 0 \tag{7.18}$$

$$x = b: \quad \beta_1 y(b) + \beta_2 y'(b) = 0 \tag{7.19}$$

where constants α_1 and α_2, and likewise β_1 and β_2, both are not zero. We shall assume that $f(x), P(x)$, and $Q(x)$ are continuous in $a \le x \le b$. The homogeneous part of equation (7.17) is

$$y'' + P(x)y' + Q(x)y = 0 \tag{7.20}$$

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of equation (7.20). Then the complementary function is given by

$$y_c = Ay_1(x) + By_2(x) \tag{7.21}$$

where A and B are two arbitrary constants. To obtain the complete general solution we vary the parameters A and B such that

$$y = A(x)y_1(x) + B(x)y_2(x) \tag{7.22}$$

which is now a complete solution of equation (7.17). Using the method of variation of parameters (see Rahman [14]), we have the following two equations:

$$\left.\begin{aligned} A'y_1 + B'y_2 &= 0 \\ A'y_1' + B'y_2' &= f(x) \end{aligned}\right\} \tag{7.23}$$

solving for A' and B', we have

$$A' = \frac{-y_2 f}{W} \quad \text{and} \quad B' = \frac{y_1 f}{W} \tag{7.24}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \text{Wronskian}$$

Let us now solve for A and B by integrating their derivatives between a and x and x and b, respectively.

The integration of these results yields, using η as dummy variable

$$\begin{aligned} A &= -\int_a^x \frac{y_2(\eta)f(\eta)}{W(\eta)}d\eta \\ B &= \int_b^x \frac{y_1(\eta)f(\eta)}{W(\eta)}d\eta \\ &= -\int_x^b \frac{y_1(\eta)f(\eta)}{W(\eta)}d\eta \end{aligned} \tag{7.25}$$

Thus, a particular solution can be obtained as

$$y = -y_1(x)\int_a^x \frac{y_2(\eta)f(\eta)}{W(\eta)}d\eta - y_2(x)\int_x^b \frac{y_1(\eta)f(\eta)}{W(\eta)}d\eta \tag{7.26}$$

Now moving $y_1(x)$ and $y_2(x)$ into the respective integrals, we have

$$y = -\left[\int_a^x \frac{y_1(x)y_2(\eta)}{W(\eta)}f(\eta)d\eta + \int_x^b \frac{y_2(x)y_1(\eta)}{W(\eta)}f(\eta)d\eta\right] \tag{7.27}$$

which is of the form

$$y = -\int_a^b G(x,\eta)f(\eta)d\eta \tag{7.28}$$

where

$$G(x,\eta) = \begin{cases} \dfrac{y_1(x)y_2(\eta)}{W(\eta)} & a \le \eta \le x, \text{i.e.} \quad \eta \le x \le b \\ \dfrac{y_2(x)y_1(\eta)}{W(\eta)} & x \le \eta \le b, \text{i.e.} \quad a \le x \le \eta. \end{cases} \tag{7.29}$$

From the way in which $y_1(x)$ and $y_2(x)$ were selected, it is clear that $G(x,\eta)$ satisfies the boundary conditions of the problem. It is also evident that $G(x,\eta)$ is a continuous function and at $x=\eta : G(\eta+0,\eta)=G(\eta-0,\eta)$. Furthermore, except at $x=\eta$, $G(x,\eta)$ satisfies the homogeneous form of the given differential equation since $y_1(x)$ and $y_2(x)$ are solutions of this equation. This can be seen as follows: Equation (7.28) must satisfy the equation (7.17) which means

$$-\int_a^b \{G_{xx} + P(x)G_x + Q(x)G\}f(\eta)d\eta = f(x)$$

The quantity under the bracket must be zero except at $x = \eta$ and thus it follows that

$$G_{xx} + P(x)G_x + Q(x)G = -\delta(\eta - x) \tag{7.30}$$

where $\delta(\eta - x)$ is a Dirac delta function.

Finally, for $\frac{\partial G}{\partial x}(x,\eta)$ we have

$$G_x(x,\eta) = \begin{cases} \dfrac{y_1'(x)y_2(\eta)}{W(\eta)} & \eta \le x \le b \\ \dfrac{y_2'(x)y_1(\eta)}{W(\eta)} & a \le x \le \eta \end{cases}$$

Hence,

$$\left.\frac{\partial G}{\partial x}\right|_{\eta+0} - \left.\frac{\partial G}{\partial x}\right|_{\eta-0} = \frac{y_1'(\eta)y_2(\eta) - y_2'(\eta)y_1(\eta)}{W(\eta)} = -1$$

which shows that $\frac{\partial G}{\partial x}(x,\eta)$ has a jump of the amount -1 at $x = \eta$. This result can be shown to be true from the relation (7.30) by integrating from $\eta + 0$ to $\eta - 0$,

$$\int_{\eta-0}^{\eta+0} G_{xx}dx + \int_{\eta-0}^{\eta+0} (P(x)G_x + Q(x)G)dx = -\int_{\eta-0}^{\eta+0} \delta(x-\eta)dx$$

$$\left.\frac{\partial G}{\partial x}\right|_{\eta+0} - \left.\frac{\partial G}{\partial x}\right|_{\eta-0} = -1$$

Because

$$\int_{\eta-0}^{\eta+0} P(x)\frac{\partial G}{\partial x}dx = P(x)\int_{\eta-0}^{\eta+0} \frac{\partial G}{\partial x}dx = P(x)[G(\eta^+,\eta) - G(\eta^-,\eta)] = 0$$

$$\text{and} \int_{\eta-0}^{\eta+0} Q(x)Gdx = Q(\eta)\int_{\eta-0}^{\eta+0} Gdx = 0,$$

since $G(x,\eta)$ is a continuous function in $\eta - 0 \le x \le \eta + 0$.

Definition 7.3

Consider, the second-order nonhomogeneous differential equation.

$$y'' + p(x)y' + Q'(x)y = f(x) \tag{7.31}$$

with the homogeneous boundary conditions

$$\left.\begin{aligned} x = a: &\quad \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ x = b: &\quad \beta_1 y(b) + \beta_2 y'(b) = 0 \end{aligned}\right\} \tag{7.32}$$

where α_1 and α_2 both are not zero, and β_1 and β_2 both are not zero.

Consider a particular solution which satisfies the differential equation (7.31) in the following form

$$y = -\int_a^b G(x,\eta)f(\eta)d\eta \tag{7.33}$$

where $G(x,\eta)$ satisfies the differential equation (7.31) such that

$$G_{xx} + P(x)G_x + Q(x)G = -\delta(\eta - x) \tag{7.34}$$

with the following properties:

(I) Boundary conditions:

$$\left.\begin{aligned} x = a: &\quad \alpha_1 G(a,\eta) + \alpha_2 G_x(a,\eta) = 0 \\ x = b: &\quad \beta_1 G(b,\eta) + \beta_2 G_x(b,\eta) = 0 \end{aligned}\right\} \tag{7.35}$$

(II) Continuity condition:

$$x = \eta, \quad G(\eta + 0,\eta) = G(\eta - 0,\eta), \tag{7.36}$$

i.e. $G(x,\eta)$ is continuous at $x = \eta$ on $a \le x \le b$.

(III) Jump discontinuity of the gradient of $G(x,\eta)$ at $x = \eta$ that means,

$$\left.\frac{\partial G}{\partial x}\right|_{x=\eta+0} - \left.\frac{\partial G}{\partial x}\right|_{x=\eta-0} = -1.$$

Then a particular solution of the given boundary value problem, i.e. equations (7.31) and (7.32) can be obtained as

$$y = -\int_a^b G(x,\eta)f(\eta)d\eta$$

where $G(x,\eta)$ is the Green's function for the boundary value problem.

Remark

It is worth mentioning here that the Green's function obtained through this procedure may or may not be symmetric, which means there is no guarantee that $G(x,\eta) = G(\eta,x)$.

Example 7.4

Find the Green's function of the homogeneous boundary value problem $y'' + y = 0$, $y(0) = 0$ and $y'(\pi) = 0$. Then solve the nonhomogeneous system

$$\begin{aligned} y'' + y &= -3 \sin 2x \\ y(0) &= 0 \\ y'(\pi) &= 0 \end{aligned}$$

Solution

The Green's function is given by

$$G(x, \eta) = \begin{cases} A \cos x + B \sin x & 0 \le x \le \eta \\ C \cos x + D \sin x & \eta \le x \le \pi \end{cases}$$

Using the boundary conditions, we have

$$\begin{aligned} x = 0: &\quad 0 = A \\ x = \pi: &\quad 0 = D \end{aligned}$$

Thus, we obtain

$$G(x, \eta) = \begin{cases} B \sin x & 0 \le x \le \eta \\ C \cos x & \eta \le x \le \pi \end{cases}$$

$G(x, \eta)$ is a continuous function at $x = \eta$. Therefore, $B \sin \eta = C \cos \eta$ from which we have $B = \alpha \cos \eta$ and $C = \alpha \sin \eta$, where α is an arbitrary constant.

The Green's function is then given by

$$G(x, \eta) = \begin{cases} \alpha \cos \eta \sin x & 0 \le x \le \eta \\ \alpha \sin \eta \cos x & \eta \le x \le \pi \end{cases}$$

This arbitrary constant α can be evaluated by the jump condition

$$\begin{aligned} \left.\frac{\partial G}{\partial x}\right|_{\eta+0} - \left.\frac{\partial G}{\partial x}\right|_{\eta-0} &= -1 \\ \text{or,} \quad \alpha[-\sin^2 \eta - \cos^2 \eta] &= -1 \end{aligned}$$

and hence $\alpha = 1$. Therefore, the required Green's function is

$$G(x, \eta) = \begin{cases} \cos \eta \sin x & 0 \le x \le \eta \\ \sin \eta \cos x & \eta \le x \le \pi \end{cases}$$

Hence the solution of the nonhomogeneous boundary value problem is

$$
\begin{aligned}
y &= -\int_0^{\pi} G(x,\eta)f(x)d\eta \\
&= -\left[\int_0^{x} G(x,\eta)f(x)d\eta + \int_x^{\pi} G(x,\eta)f(x)d\eta\right] \\
&= -\left[\int_0^{x} \sin\eta\cos x(-3\sin 2\eta)d\eta + \int_x^{\pi} \sin x\cos\eta(-3\sin 2\eta)d\eta\right] \\
&= 3\left[\cos x\int_0^{x} \sin\eta\sin 2\eta d\eta + \sin x\int_x^{\pi} \cos\eta\sin 2\eta d\eta\right]
\end{aligned}
$$

Performing the indicated integration, we obtain

$$
\begin{aligned}
y &= 3\left[\tfrac{2}{3}\right][\cos x\sin^3 x + \sin x(1+\cos^3 x)] \\
&= 2\sin x + \sin 2x
\end{aligned}
$$

Using the elementary operator method, the solution of the nonhomogeneous ODE is given by

$$y = A\cos x + B\sin x + \sin 2x$$

With the given boundary conditions, we obtain $A = 0$ and $B = 0$, and hence the solution is

$$y = 2\sin x + \sin 2x$$

which is identical to the Green's function method.

Example 7.5

Find a particular solution of the following boundary value problem by using the Green's function method:

$$y'' = -x; \quad y(0) = 0, y(1) = 0$$

Solution

The Green's function is obtained from the homogeneous part of the differential equation and is given by,

$$G(x,\eta) = \begin{cases} Ax + B & 0 \le x \le \eta \\ \alpha x + \beta & \eta \le x \le 1 \end{cases}$$

Using the boundary conditions, we obtain
$0 = B$ and $\alpha = -\beta$ and hence

$$G(x,\eta) = \begin{cases} Ax & 0 \leq x \leq \eta \\ \beta(1-x) & \eta \leq x \leq 1 \end{cases}$$

Now continuity condition yields $A\eta = \beta(1-\eta)$ which can be written as $\frac{A}{1-\eta} = \frac{\beta}{\eta} = \gamma$ such that $A = \gamma(1-\eta)$, where γ is the arbitrary constant. Hence we have

$$G(x,\eta) = \begin{cases} \gamma(1-\eta)x & 0 \leq x \leq \eta \\ \gamma(1-x)\eta & \eta \leq x \leq 1 \end{cases}$$

From the jump condition at $x = \eta$, we obtain

$$\left.\frac{\partial G}{\partial x}\right|_{\eta+0} - \left.\frac{\partial G}{\partial x}\right|_{\eta-0} = -1$$
$$or, \quad \gamma[-\eta - (1-\eta)] = -1$$

such that $\gamma = 1$.
Hence the Green's function is determined as

$$G(x,\eta) = \begin{cases} (1-\eta)x & 0 \leq x \leq \eta \\ (1-x)\eta & \eta \leq x \leq 1 \end{cases}$$

Changing the roles of x and η, we have

$$G(x,\eta) = G(\eta,x) = \begin{cases} (1-x)\eta & 0 \leq \eta \leq x \\ (1-\eta)x & x \leq \eta \leq 1 \end{cases}$$

Thus, a particular solution of this boundary problem is obtained as:

$$\begin{aligned} y(x) &= -\int_0^1 G(x,\eta)f(\eta) \\ &= -\left[\int_0^x G(x,\eta)f(\eta)d\eta + \int_x^1 G(x,\eta)f(\eta)d\eta\right] \\ &= -\left[\int_0^x (1-x)\eta(-\eta)d\eta + \int_x^1 (1-\eta)x(-\eta)d\eta\right] \\ &= \left[(1-x)\left|\frac{\eta^3}{3}\right|_0^x + x\left|\frac{\eta^2}{2} - \frac{\eta^3}{3}\right|_x^1\right] \\ &= \tfrac{x^3}{3} + \tfrac{x}{6} - \tfrac{x^3}{2} = \tfrac{x}{6}(1-x^2) \end{aligned}$$

7.3 Green's function in two-dimensions

We already saw the application of Green's function in one-dimension to boundary value problems in ordinary differential equations in the previous section. In this section, we illustrate the use of Green's function in two-dimensions to the boundary value problems in partial differential equations arising in a wide class of problems in engineering and mathematical physics.

As Green's function is always associated with the Dirac delta function, it is therefore useful to formally define this function. We define the Dirac delta function $\delta(x-\xi, y-\eta, z-\zeta)$ in two-dimensions by

$$\text{I.}\quad \delta(x-\xi, y-\eta, z-\zeta) = \begin{cases} \infty, & x=\xi, y=\eta \\ 0 & \text{otherwise} \end{cases} \tag{7.37}$$

$$\text{II.}\quad \iint_{R_\varepsilon} \delta(x-\xi, y-\eta)dxdy = 1, R_\varepsilon : (x-\xi)^2 + (y-\eta)^2 < \varepsilon^2 \tag{7.38}$$

$$\text{III.}\quad \iint_R f(x,y)\delta(x-\xi, y-\eta)dxdy = f(\xi,\eta) \tag{7.39}$$

for arbitrary continuous function $f(x,y)$ in the region R.

Remark

The Dirac delta function is not a regular function but it is a generalized function as defined by Lighthill [9] in his book "Fourier Analysis and Generalized Function", Cambridge University Press, 1959. This function is also called an impulse function which is defined as the limit for a sequence of regular well-behaved functions having the required property that the area remains a constant (unity) as the width is reduced. The limit of this sequence is taken to define the impulse function as the width is reduced toward zero. For more information and an elegant treatment of the delta function as a generalized function, interested readers are referred to "Theory of Distribution" by L. Schwartz (1843–1921).

If $\delta(x-\xi)$ and $\delta(y-\eta)$ are one-dimensional delta functions, then we have

$$\iint_R f(x,y)\delta(x-\xi)\delta(y-\eta)dxdy = f(\xi,\eta) \tag{7.40}$$

Since equations (7.39) and (7.40) hold for an arbitrary continuous function f, we conclude that

$$\delta(x-\xi, y-\eta) = \delta(x-\xi)\delta(y-\eta) \tag{7.41}$$

which simply implies that the two-dimensional delta function is the product of one-dimensional delta functions. Higher dimensional delta functions can be defined in a similar manner.

7.3.1 Two-dimensional Green's function

The application of Green's function in two-dimension can best be described by considering the solution of the Dirichlet problem.

$$\nabla^2 u = h(x,y) \qquad \text{in the two-dimensional region R} \tag{7.42}$$

$$u = f(x,y) \qquad \text{on the boundary C} \tag{7.43}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Before attempting to solve this boundary value problem heuristically, we first define the Green's function for the Laplace operator. Then, the Green's function for the Dirichlet problem involving the Helmholtz operator may be defined in a similar manner.

The Green's function denoted by $G(x,y;\xi,\eta)$ for the Dirichlet problem involving the Laplace operator is defined as the function which satisfies the following properties:

(i) $$\nabla^2 G = \delta(x-\xi, y-\eta) \quad \text{in } R \tag{7.44}$$

$$G = 0 \quad \text{on } C \tag{7.45}$$

(ii) G is symmetric, that is,

$$G(x,y;\xi,\eta) = G(\xi,\eta;x,y) \tag{7.46}$$

(iii) G is continuous in $x,y;\xi,\eta$ but $\frac{\partial G}{\partial n}$ the normal derivative has a discontinuity at the point (ξ,η) which is specified by the equation

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} \frac{\partial G}{\partial n} ds = 1 \tag{7.47}$$

where n is the outward normal to the circle

$$C_\varepsilon : \quad (x-\xi)^2 + (y-\eta)^2 = \varepsilon^2$$

Remark

The Green's function G may be interpreted as the response of the system at a field point (x,y) due to a delta function $\delta(x,y)$ input at the source point (ξ,η). G is continuous everywhere in the region R, and its first and second derivatives are continuous in R except at (ξ,η). Thus, the property (i) essentially states that $\nabla^2 G = 0$ everywhere except at the source point (ξ,η).

Properties (ii) and (iii) pertaining to the Green's function G can be established directly by using the Green's second identity of vector calculus [1]. This formula

states that if ϕ and ψ are two functions having continuous second partial derivatives in a region R bounded by a curve C, then

$$\iint_R (\phi\nabla^2\psi - \psi\nabla^2\phi)dS = \int_C \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) ds \tag{7.48}$$

where dS is the elementary area and ds the elementary length.

Now, let us consider that $\phi = G(x, y; \xi; \eta)$ and $\psi = G(x, y; \xi^*\eta^*)$ then from equation (7.48)

$$\begin{aligned}&\iint_R \{G(x, y; \xi, \eta)\nabla^2 G(x, y; \xi^*\eta^*) - G(x, y; \xi^*, \eta^*)\nabla^2 G(x, y; \xi, \eta)\}dS \\ &= \int_C \left\{G(x, y; \xi, \eta)\frac{\partial G}{\partial n}(x, y; \xi^*, \eta^*) - G(x, y; \xi^*, \eta^*)\frac{\partial G}{\partial n}(x, y; \xi, \eta)\right\} ds\end{aligned} \tag{7.49}$$

$$\text{Since} \quad G(x, y; \xi, \eta) = 0 \quad \text{on } C$$

$$\text{and} \quad G(x, y; \xi^*, \eta^*) = 0 \quad \text{on } C$$

$$\text{also} \quad \nabla^2 G(x, y; \xi^*, \eta^*) = \delta(x - \xi, y - \eta) \quad \text{in } R$$

$$\text{and} \quad \nabla^2 G(x, y; \xi^*, \eta^*) = \delta(x - \xi^*, y - \eta^*) \quad \text{in } R$$

Hence we obtain from equation (7.49)

$$\iint_R G(x, y; \xi, \eta)\delta(x - \xi^*, y - \eta^*)dxdy = \iint_R G(x, y; \xi^*, \eta^*)\delta(x, \xi; y - \eta)dxdy$$

which reduces to

$$G(\xi^*, \eta^*; \xi, \eta) = G(\xi, \eta; \xi^*, \eta^*)$$

Thus, the Green's function is symmetric.

To prove property (iii) of Green's function, we simply integrate both sides of equation (7.44) which yields

$$\iint_{R_\varepsilon} \nabla^2 G dS = \iint_{R_\varepsilon} \delta(x - \xi, y - \eta)dxdy = 1$$

where R_ε is the region bounded by the circle C_ε.
Thus, it follows that

$$\lim_{\varepsilon\to 0} \iint_{R_\varepsilon} \nabla^2 G dS = 1$$

and using the Divergence theorem [1],

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{\partial G}{\partial n} ds = 1.$$

Theorem 7.1

Prove that the solution of the Dirichlet problem

$$\nabla^2 u = h(x,y) \quad in\, R \tag{7.50}$$

subject to the boundary conditions

$$u = f(x,y) \quad on\, C \tag{7.51}$$

is given by

$$u(x,y) = \iint_R G(x,y;\xi,\eta)h(\xi,\eta)d\xi d\eta + \int_C f\frac{\partial G}{\partial n}ds \tag{7.52}$$

where G is the Green's function and n denotes the outward normal to the boundary C of the region R. It can be easily seen that the solution of the problem is determined if the Green's function is known.

Proof

In equation (7.48), let us consider that

$$\phi(\xi,\eta) = G(\xi,\eta;x,y) \quad \text{and} \quad \psi(\xi,\eta) = u(\xi,\eta),$$

and we obtain

$$\begin{aligned}&\iint_R [G(\xi,\eta;x,y)\nabla^2 u - u(\xi,\eta)\nabla^2 G]d\xi d\eta \\ &= \int_C \left[G(\xi,\eta;x,y)\frac{\partial u}{\partial n} - u(\xi,\eta)\frac{\partial G}{\partial n}\right]ds\end{aligned} \tag{7.53}$$

Here, $\nabla^2 u = h(\xi,\eta)$ and $\nabla^2 G = \delta(\xi - x, \eta - y)$ in the region R.
Thus, we obtain from equation (7.53)

$$\begin{aligned}&\iint_R \{G(\xi,\eta;x,y)h(\xi,\eta) - u(\xi,\eta)\delta(\xi - x,\eta - y)\}d\xi d\eta \\ &= \int_C \left\{G(\xi,\eta;x,y)\frac{\partial u}{\partial n} - u(\xi,\eta)\frac{\partial G}{\partial n}\right\}ds\end{aligned} \tag{7.54}$$

Since $G = 0$ and $u = f$ on C, and noting that G is symmetric, it follows that

$$u(x,y) = \iint_R G(x,y;\xi,\eta)h(\xi,\eta)d\xi d\eta + \int_C f\frac{\partial G}{\partial n}ds$$

which is the required proof as asserted.

7.3.2 Method of Green's function

It is often convenient to seek G as the sum of a particular integral of the non-homogeneous differential equation and the solution of the associated homogeneous differential equation. That is, G may assume the form

$$G(\xi,\eta;x,y) = g_h(\xi,\eta;x,y) + g_p(\xi,\eta;x,y) \tag{7.55}$$

where g_h, known as the free-space Green's function, satisfies

$$\nabla^2 g_h = 0 \quad \text{in } R \tag{7.56}$$

and g_p satisfies

$$\nabla^2 g_p = \delta(\xi - x, \eta - y) \quad \text{in } R \tag{7.57}$$

so that by superposition $G = g_h + g_p$ satisfies

$$\nabla^2 G = \delta(\xi - x, \eta - y) \tag{7.58}$$

Note that (x,y) will denote the source point and (ξ,η) denotes the field point. Also $G = 0$ on the boundary requires that

$$g_h = -g_p \tag{7.59}$$

and that g_p need not satisfy the boundary condition.

In the following, we will demonstrate the method to find g_p for the Laplace and Helmholtz operators.

7.3.3 The Laplace operator

In this case, g_p must satisfy

$$\nabla^2 g_p = \delta(\xi - x, \eta - y) \qquad \text{in } R$$

Then for $r = \sqrt{(\xi - x)^2 + (\eta - y)^2} > 0$, that is, for $\xi \neq x$, and $\eta \neq y$, we have by taking (x,y) as the centre (assume that g_p is independent of θ)

$$\nabla^2 g_p = 0, \quad \text{or,} \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g_p}{\partial r}\right) = 0.$$

The solution of which is given by

$$g_p = \alpha + \beta \ln r$$

Now, apply the condition

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\partial g_p}{\partial n}ds = \lim_{\varepsilon\to 0}\int_0^{2\pi}\left(\frac{\beta}{r}\right)rd\theta = 1$$

Thus, $\beta = \frac{1}{2\pi}$ and α is arbitrary. For simplicity, we choose $\alpha = 0$. Then g_p takes the form

$$g_p = \tfrac{1}{2\pi}\ln r \tag{7.60}$$

This is known as the fundamental solution of the Laplace's equation in two-dimension.

7.3.4 The Helmholtz operator

Here, g_p satisfies the following equation

$$(\nabla^2 + \lambda^2)g_p = \delta(\xi - x, \eta - y) \tag{7.61}$$

Again for $r > 0$, we find

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g_p}{\partial r}\right) + \lambda^2 g_p = 0$$

$$\text{or}\quad r^2(g_p)_{rr} + r(g_p)_r + \lambda^2 r^2 g_p = 0 \tag{7.62}$$

This is the Bessel equation of order zero, the solution of which is

$$g_p = \alpha J_0(\lambda r) + \beta Y_0(\lambda r) \tag{7.63}$$

Since the behaviour of J_0 at $r = 0$ is not singular, we set $\alpha = 0$. Thus, we have

$$g_p = \beta Y_0(\lambda r)$$

But for very small r, $Y_0 \approx \frac{2}{\pi}\ln r$.
Applying the condition, $\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\partial g_p}{\partial n}ds = 1$, we obtain

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\beta\frac{\partial Y_0}{\partial r}ds = 1$$

$$\lim_{\varepsilon\to 0}\int_0^{2\pi}\beta\left(\frac{2}{\pi}\right)\left(\frac{1}{r}\right)rd\theta = 1$$

$$\tfrac{2\beta}{\pi}(2\pi) = 1$$

$$\beta = \tfrac{1}{4}$$

Thus,

$$g_p = \tfrac{1}{4} Y_0(\lambda r) \tag{7.64}$$

Since $(\nabla^2 + \lambda^2) \to \nabla^2$ as $\lambda \to 0$, it follows that $\frac{1}{4} Y_0(\lambda r) \to \frac{1}{2\pi} \ln r$ as $\lambda \to 0$.

Theorem 7.2: (Solution of Dirichlet problem using the Laplace operator)

Show that the method of Green's function can be used to obtain the solution of the Dirichlet problem described by the Laplace operator:

$$\nabla^2 u = h \quad in\ R \tag{7.65}$$

$$u = f \quad on\ C \tag{7.66}$$

and the solution is given by

$$u(x,y) = \iint_R G(x,y;\xi,\eta) h(\xi,\eta) d\xi d\eta + \int_C f \frac{\partial G}{\partial n} ds \tag{7.67}$$

where R is a circular region with boundary C.

Proof

Applying Green's second formula

$$\iint_R (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dS = \int_C \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds \tag{7.68}$$

to the functions $\phi(\xi,\eta) = G(\xi,\eta;x,y)$ and $\psi(\xi,\eta) = u(\xi,\eta)$, we obtain

$$\iint_R (G \nabla^2 u - u \nabla^2 G) dS = \int_C \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right) ds$$

$$\text{But} \qquad \nabla^2 u = h(\xi,\eta)$$

$$\text{and} \qquad \nabla^2 G = \delta(\xi - x, \eta - y) \quad \text{in } R$$

Thus, we have

$$\iint_R \{G(\xi,\eta;x,y) h(\xi,\eta) - u(\xi,\eta) \delta(\xi - x, \eta - y)\} dS$$
$$= \int_C \left\{ G(\xi,\eta;x,y) \frac{\partial u}{\partial n} - u(\xi,\eta) \frac{\partial G}{\partial n} \right\} ds \tag{7.69}$$

Because $G = 0$ and $u = f$ on the boundary C, and G is symmetric, it follows that

$$u(x,y) = \iint_R G(x,y;\xi,\eta)h(\xi,\eta)d\xi d\eta + \int_C f\frac{\partial G}{\partial n}ds,$$

which is the required proof as asserted.

Example 7.6

Consider the Dirichlet problem for the unit circle given by

$$\nabla^2 u = 0 \qquad in\ R \tag{7.70}$$

$$u = f(\theta) \qquad on\ C \tag{7.71}$$

Find the solution by using the Green's function method.

Solution

We introduce the polar coordinates by means of the equations

$$x = \rho\cos\theta \qquad \xi = \sigma\cos\beta$$

$$y = \rho\sin\theta \qquad \eta = \sigma\sin\beta$$

so that

$$r^2 = (x-\xi)^2 + (y-\eta)^2 = \sigma^2 + \rho^2 - 2\rho\sigma\cos(\beta-\theta).$$

To find the Green's function, let us consider $G(x,y)$ is the solution of the sum of two solutions, one regular and the other one singular. We know G satisfies

$$\nabla^2 G = \delta(\xi - x, \eta - y)$$

$$\text{and if } G = g_h + g_p,$$

$$\text{then } \nabla^2 g_p = \delta(\xi - x, \eta - y)$$

$$\text{and } \nabla^2 g_h = 0.$$

These equations in polar coordinates (see Figure 7.3) can be written as

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_p}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 g_p}{\partial\theta^2} = \delta(\sigma-\rho,\beta-\theta) \tag{7.72}$$

$$\text{and} \quad \frac{1}{\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial g_h}{\partial\sigma}\right) + \frac{1}{\sigma^2}\frac{\partial^2 g_h}{\partial\beta^2} = 0 \tag{7.73}$$

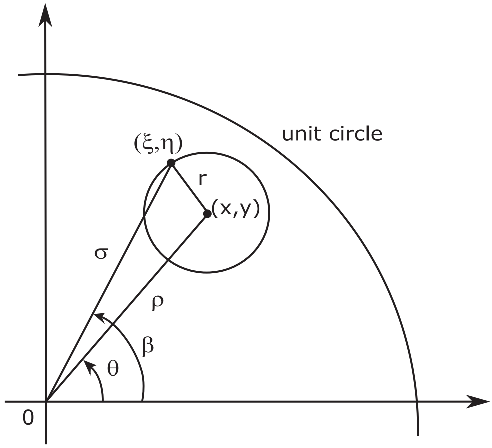

Figure 7.3: A unit circle without an image point.

By the method of separation of variables, the solution of equation (7.73) can be written as (see Rahman [13, 15])

$$g_h = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sigma^n (a_n \cos n\beta + b_n \sin n\beta) \tag{7.74}$$

A singular solution for equation (7.72) is given by

$$g_p = \tfrac{1}{4\pi} \ln r^2 = \tfrac{1}{4\pi} \ln [\sigma^2 + \rho^2 - 2\rho\sigma \cos(\beta - \theta)]$$

Thus, when $\sigma = 1$ on the boundary C,

$$g_h = -g_p = -\tfrac{1}{4\pi} \ln[1 + \rho^2 - 2\rho \cos(\beta - \theta)]$$

The relation

$$\ln [1 + \rho^2 - 2\rho \cos(\beta - \theta)] = -2 \sum_{n=1}^{\infty} \frac{\rho^n \cos n(\beta - \theta)}{n} \tag{7.75}$$

can be established as follows:

$$\begin{aligned}
\ln[1 + \rho^2 - \rho(e^{i(\beta-\theta)} + e^{-i(\beta-\theta)})] &= \ln\{(1 - \rho e^{i(\beta-\theta)})(1 - \rho e^{-i(\beta-\theta)})\} \\
&= \ln(1 - \rho e^{i(\beta-\theta)}) + \ln(1 - \rho e^{-i(\beta-\theta)}) \\
&= -[\rho e^{i(\beta-\theta)} + \tfrac{\rho^2}{2} e^{2i(\beta-\theta)} + \cdots] +
\end{aligned}$$

$$-[\rho e^{-i(\beta-\theta)} - \tfrac{\rho^2}{2} e^{-2i(\beta-\theta)} + \cdots]$$

$$= -2\sum_{n=1}^{\infty} \frac{\rho^n \cos n(\beta-\theta)}{n}$$

When $\sigma = 1$ at the circumference of the unit circle, we obtain

$$\frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{\rho^n \cos n(\beta-\theta)}{n} = \sum_{n=1}^{\infty} a_n \cos n\beta + b_n \sin n\beta$$

Now equating the coefficients of $\cos n\beta$ and $\sin n\beta$ to determine a_n and b_n, we find

$$a_n = \frac{\rho^n}{2\pi n} \cos n\theta$$

$$b_n = \frac{\rho^n}{2\pi n} \sin n\theta$$

It therefore follows that equation (7.74) becomes

$$g_h(\rho,\theta;\sigma,\beta) = \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{(\rho\sigma)^n}{n} \cos n(\beta-\theta)$$

$$= -\tfrac{1}{4\pi} \ln[1 + (\rho a)^2 - 2(\rho\sigma)\cos(\beta-\theta)]$$

Hence the Green's function for this problem is

$$G(\rho,\theta;\sigma,\beta) = g_p + g_h$$

$$= \tfrac{1}{4\pi} \ln[\sigma^2 + \rho^2 - 2\sigma\rho\cos(\beta-\theta)]$$

$$- \tfrac{1}{4\pi} \ln[1 + (\rho\sigma)^2 - 2(\rho\sigma)\cos(\beta-\theta)] \qquad (7.76)$$

from which we find

$$\left.\frac{\partial G}{\partial n}\right|_{\text{on } C} = \left(\frac{\partial G}{\partial \sigma}\right)_{\sigma=1} = \frac{1}{2\pi}\left[\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\beta-\theta)}\right]$$

If $h = 0$, the solution of the problem reduces to

$$u(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\beta-\theta)} f(\beta)d\beta$$

which is the Poisson's Integral Formula.

Theorem 7.3: (Solution of Dirichlet problem using the Helmholtz operator)

Show that the Green's function method can be used to solve the Dirichlet problem for the Helmholtz operator:

$$\nabla^2 u + \lambda^2 u = h \quad \text{in } R \tag{7.77}$$

$$u = f \quad \text{on } C \tag{7.78}$$

where R is a circular region of unit radius with boundary C.

Proof

The Green's function must satisfy the Helmholtz equation in the following form

$$\nabla^2 G + \lambda^2 G = \delta(\xi - x, \eta - y) \quad \text{in } R$$

$$G = 0 \quad \text{on } R \tag{7.79}$$

We seek the solution in the following form

$$G(\xi, \eta; x, y) = g_h(\xi, \eta; x, y) + g_p(\xi, \eta; x, y)$$

$$\text{such that} \quad (\nabla^2 + \lambda^2) g_h = 0 \tag{7.80}$$

$$\text{and} \quad (\nabla^2 + \lambda^2) g_p = \delta(\xi - x, \eta - y) \tag{7.81}$$

The solution of equation (7.81) yields equation (7.63)

$$g_p = \frac{1}{4} Y_0(\lambda r) \tag{7.82}$$

where

$$r = [(\xi - x)^2 + (\eta - y)^2]^{\frac{1}{2}}$$

The solution for equation (7.80) can be determined by the method of separation of variables. Thus, the solution in polar coordinates as given below

$$x = \rho\cos\theta \qquad \xi = \sigma\cos\beta$$

$$y = \rho\sin\theta \qquad \eta = \sigma\sin\beta$$

may be written in the form

$$g_h(\rho, \theta; \sigma, \beta) = \sum_{n=0}^{\infty} J_n(\lambda\sigma)[a_n \cos n\beta + b_n \sin n\beta] \tag{7.83}$$

But on the boundary C,

$$g_h + g_p = 0$$

Therefore,

$$g_h = -g_p = -\tfrac{1}{4}Y_0(\lambda r)$$

where $$r = [\rho^2 + \sigma^2 - 2\rho\sigma\cos(\beta - \theta)]^{\frac{1}{2}}$$

and at $\sigma = 1$, $$r = [1 + \rho^2 - 2\rho\cos(\beta - \theta)]^{\frac{1}{2}}$$

Thus, on the boundary ($\sigma = 1$), these two solutions yield

$$-\frac{1}{4}Y_0(\lambda r) = \sum_{n=0}^{\infty} J_n(\lambda)[a_n \cos n\beta + b_n \sin n\beta]$$

which is a Fourier expansion. The Fourier coefficients are obtained as

$$a_0 = -\frac{1}{8\pi J_0(\lambda)}\int_{-\pi}^{\pi} Y_0\left[\lambda\sqrt{1 + \rho^2 - 2\rho\cos(\beta - \theta)}\right]d\beta$$

$$a_n = -\frac{1}{4\pi J_0(\lambda)}\int_{-\pi}^{\pi} Y_0\left[\lambda\sqrt{1 + \rho^2 - 2\rho\cos(\beta - \theta)}\right]\cos n\beta d\beta$$

$$b_n = -\frac{1}{4\pi J_0(\lambda)}\int_{-\pi}^{\pi} Y_0\left[\lambda\sqrt{1 + \rho^2 - 2\rho\cos(\beta - \theta)}\right]\sin n\beta d\beta$$

$$n = 1, 2, 3, \ldots$$

From the Green's theorem, we have

$$\iint_R \{G(\nabla^2 + \lambda^2)u - u(\nabla^2 + \lambda^2)G\}dS = \int_C \left\{G\left(\frac{\partial u}{\partial n}\right) - u\left(\frac{\partial G}{\partial n}\right)\right\}ds$$

But we know $(\nabla^2 + \lambda^2)G = \delta(\xi - x, \eta - y)$, and $G = 0$ on C, and $(\nabla^2 + \lambda^2)u = h$. Therefore,

$$u(x, y) = \iint_R h(\xi, \eta)G(\xi, \eta; x, y)d\xi d\eta + \int_C f(\xi, \eta)\frac{\partial G}{\partial n}ds$$

where G is given by

$$\begin{aligned} G(\xi, \eta; x, y) &= g_p + g_h \\ &= \frac{1}{4}Y_0(\lambda r) + \sum_{n=0}^{\infty} J_n(\lambda\sigma)\{a_n \cos n\beta + b_n \sin n\beta\} \end{aligned}$$

7.3.5 To obtain Green's function by the method of images

The Green's function can be obtained by using the method of images. This method is based essentially on the construction of Green's function for a finite domain. This method is restricted in the sense that it can be applied only to certain class of problems with simple boundary geometry.

Let us consider the Dirichlet problem to illustrate this method.

Let $P(\xi,\eta)$ be the field point in the circle R, and let $Q(x,y)$ be the source point also in R. The distance between P and Q is $r=\sqrt{\sigma^2+\rho^2-2\rho\sigma\cos(\beta-\theta)}$. Let Q' be the image which lies outside of R on the same ray from the origin opposite to the source point Q as shown in Figure 7.4 such that

$$(OQ)(OQ')=\sigma^2$$

where σ is the radius of the circle through the point P centered at the origin.

Since the two triangles OPQ and OPQ' are similar by virtue of the hypothesis $(OQ)(OQ')=\sigma^2$ and by possessing the common angle at O, we have

$$\frac{r}{r'}=\frac{\sigma}{\rho} \tag{7.84}$$

where $r'=PQ'$ and $\rho=OQ$. If $\sigma=1$, then equation (7.84) becomes

$$\frac{r}{r'}\frac{1}{\rho}=1.$$

Then taking the logarithm of both sides with a multiple of $\frac{1}{2\pi}$, we obtain

$$\frac{1}{2\pi}\ln\left(\frac{r}{r'\rho}\right)=0$$

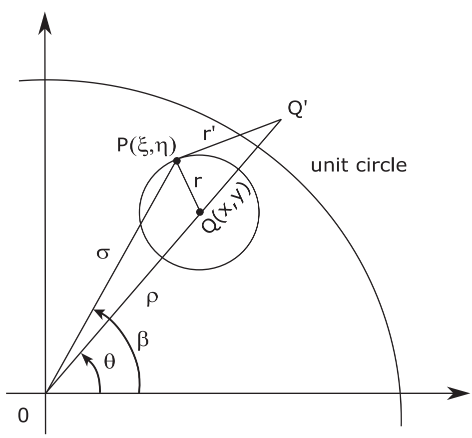

Figure 7.4: A unit circle with an image point.

$$\text{or} \quad \frac{1}{2\pi}\ln r - \frac{1}{2\pi}\ln r' + \frac{1}{2\pi}\ln\frac{1}{\rho} = 0 \tag{7.85}$$

This equation signifies that $\frac{1}{2\pi}\ln\left(\frac{r'}{r\rho}\right)$ is harmonic in R except at Q and satisfies the Laplace's equation

$$\nabla^2 G = \delta(\xi - x, \eta - y) \tag{7.86}$$

Note that $\ln r'$ is harmonic everywhere except at Q', which is outside the domain R. This suggests that we can choose the Green's function as

$$G = \frac{1}{2\pi}\ln r - \frac{1}{2\pi}\ln r' + \frac{1}{2\pi}\ln\frac{1}{\rho} \tag{7.87}$$

Given that Q' is at $(\frac{1}{\rho}, \theta)$, G in polar coordinates takes the form

$$\begin{aligned} G(\rho, \theta; \sigma, \beta) &= \frac{1}{4\pi}\ln\left(\sigma^2 + \rho^2 - 2\rho\sigma\cos(\beta - \theta)\right) \\ &\quad - \frac{1}{4\pi}\ln\left(\frac{1}{\sigma^2} + \rho^2 - \frac{2\rho}{\sigma}\cos(\beta - \theta)\right) + \frac{1}{2\pi}\ln\frac{1}{\sigma} \end{aligned} \tag{7.88}$$

which is the same as before.

Remark

Note that in the Green's function expression in equation (7.87) or equation (7.88), the first term represents the potential due to a unit line charge at the source point, whereas the second term represents the potential due to negative unit charge at the image point and the third term represents a uniform potential. The sum of these potentials makes up the potential field.

Example 7.7

Find the solution of the following boundary value problem by the method of images:

$$\nabla^2 u = h \quad \text{in } \eta > 0$$

$$u = f \quad \text{on } \eta = 0$$

Solution

The image point should be obvious by inspection. Thus, if we construct

$$G = \frac{1}{4\pi}\ln[(\xi - x)^2 + (\eta - y)^2] - \frac{1}{4\pi}\ln[(\xi - x)^2 + (\eta + y)^2] \tag{7.89}$$

the condition that $G = 0$ on $\eta = 0$ is clearly satisfied. Also G satisfies

$$\nabla^2 G = \delta(\xi - x, \eta - y)$$

and with $\frac{\partial G}{\partial n}|_C = [-\frac{\partial G}{\partial \eta}]_{\eta=0}$, the solution is given by

$$\begin{aligned}
u(x,y) &= \iint_R G(x,y;\xi,\eta)h(\xi,\eta)d\xi d\eta + \int_C f\frac{\partial G}{\partial n}ds \\
&= \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(\xi-x)^2+y^2} + \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\ln\left\{\frac{(\xi-x)^2+(\eta-y)^2}{(\xi-x)^2+(\eta+y)^2}\right\} \\
&\quad \times h(\xi,\eta)d\xi d\eta
\end{aligned}$$

7.3.6 Method of eigenfunctions

Green's function can also be obtained by applying the method of eigenfunctions. We consider the boundary value problem

$$\begin{aligned}
\nabla^2 u &= h \quad \text{in } R \\
u &= f \quad \text{on } C
\end{aligned} \tag{7.90}$$

The Green's function must satisfy

$$\begin{aligned}
\nabla^2 G &= \delta(\xi-x,\eta-y) \quad \text{in } R \\
G &= 0 \qquad \text{on } C
\end{aligned} \tag{7.91}$$

and hence the associated eigenvalue problem is

$$\begin{aligned}
\nabla^2\phi + \lambda\phi &= 0 \quad \text{in } R \\
\phi &= 0 \quad \text{on } C
\end{aligned} \tag{7.92}$$

Let ϕ_{mn} be the eigenfunctions and λ_{mn} be the corresponding eigenvalues. We then expand G and δ in terms of the eigenfunctions ϕ_{mn}

$$G(\xi,\eta;x,y) = \sum_m\sum_n a_{mn}(x,y)\phi(\xi,\eta) \tag{7.93}$$

$$\delta(\xi,\eta;x,y) = \sum_m\sum_n b_{mn}(x,y)\phi_{mn}(\xi,\eta) \tag{7.94}$$

where

$$\begin{aligned}
b_{mn} &= \frac{1}{||\phi_{mn}||^2}\iint_R \delta(\xi,\eta;x,y)\phi_{mn}(\xi,\eta)d\xi d\eta \\
&= \frac{\phi_{mn}(x,y)}{||\phi_{mn}||^2}
\end{aligned} \tag{7.95}$$

in which $||\phi_{mn}||^2 = \iint_R \phi_{mn}^2 d\xi d\eta$. Now, substituting equations (7.93) and (7.94) into equation (7.91) and using the relation from equation (7.92),

$$\nabla^2\phi_{mn} + \lambda_{mn}\phi_{mn} = 0$$

we obtain

$$-\sum_m\sum_n \lambda_{mn}a_{mn}(x,y)\phi_{mn}(\xi,\eta) = \sum_m\sum_n \frac{\phi_{mn}(x,y)\phi_{mn}(\xi,\eta)}{||\phi_{mn}||^2}$$

Therefore, we have

$$a_{mn}(x,y) = \frac{-\phi_{mn}(x,y)}{\lambda_{mn}||\phi_{mn}||^2} \tag{7.96}$$

Thus, the Green's function is given by

$$G(\xi,\eta;x,y) = -\sum_m\sum_n \frac{\phi_{mn}(x,y)\phi_{mn}(\xi,\eta)}{\lambda_{mn}||\phi_{mn}||^2} \tag{7.97}$$

We shall demonstrate this method by the following example.

Example 7.8

Find the Green's function for the following boundary value problem

$$\nabla^2 u = h \quad \text{in } R$$

$$u = 0 \quad \text{on } C$$

Solution

The eigenfunction can be obtained explicitly by the method of separation of variables.

We assume a solution in the form

$$\phi(\xi,\eta) = X(\xi)Y(\eta)$$

Substitution of this into $\nabla^2\phi + \lambda\phi = 0$ in R and $\phi = 0$ on C, yields

$$X'' + \alpha^2 X = 0$$

$$Y'' + (\lambda - \alpha^2)Y = 0$$

where α^2 is a separation constant. With the homogeneous boundary conditions $X(0) = X(a) = 0$ and $Y(0) = Y(b) = 0, X$ and Y are found to be

$$X_m(\xi) = A_m \sin\tfrac{m\pi\xi}{a} \quad \text{and} \quad Y_n(\eta) = B_n \sin\tfrac{n\pi\eta}{b}$$

We then have

$$\lambda_{mn} = \pi^2 \left(\tfrac{m^2}{a^2} + \tfrac{n^2}{b^2}\right) \quad \text{with} \quad \alpha = \tfrac{m\pi}{a}$$

Thus, the eigenfunctions are given by

$$\phi_{mn}(\xi, \eta) = \sin \tfrac{m\pi\xi}{a} \sin \tfrac{n\pi\eta}{b}$$

Hence

$$\begin{aligned} ||\phi_{mn}||^2 &= \int_0^a \int_0^b \sin^2 \frac{m\pi\xi}{a} \sin^2 \frac{n\pi\eta}{b} d\xi d\eta \\ &= \tfrac{ab}{4} \end{aligned}$$

so that the Green's function can be obtained as

$$G(\xi, \eta; x, y) = -\frac{4ab}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}}{(m^2b^2 + a^2n^2)}$$

7.4 Green's function in three-dimensions

Since most of the problems encountered in the physical sciences are in three-dimension, we can extend the Green's function to three or more dimensions. Let us extend our definition of Green's function in three-dimensions.

Consider the Dirichlet problem involving Laplace operator. The Green's function satisfies the following:

(i)
$$\left.\begin{aligned} \nabla^2 G &= \delta(x-\xi, y-\eta, z-\zeta) \quad \text{in } R \\ G &= \quad 0 \quad \text{on } S. \end{aligned}\right\} \tag{7.98}$$

(ii)
$$G(x,y,z;\xi,\eta,\zeta) = G(\xi,\eta,\zeta;x,y,z) \tag{7.99}$$

(iii)
$$\lim_{\varepsilon\to 0} \iint_{S_\varepsilon} \frac{\partial g}{\partial n} ds = 1 \tag{7.100}$$

where n is the outward normal to the surface
$S_\varepsilon : (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = \varepsilon^2$

Proceeding as in the two-dimensional case, the solution of Dirichlet problem

$$\left.\begin{aligned} \nabla^2\phi &= h \quad \text{in } R \\ \phi &= f \quad \text{on } S \end{aligned}\right\} \tag{7.101}$$

$$\text{is} \qquad u(x,y,z) = \iiint_R GhdR + \iint_S fG_n dS \tag{7.102}$$

To obtain the Green's function, we let the Green's function to have two parts as

$$G(\xi,\eta,\zeta;x,y,z) = g_h(\xi,\eta,\zeta;x,y,z) + g_p(\xi,\eta,\zeta;x,y,z)$$

$$\text{where} \quad \nabla^2 g_h = \delta(x-\xi, y-\eta, z-\zeta) \quad \text{in } R$$

$$\text{and} \quad \nabla^2 g_p = 0 \quad \text{on } S$$

$$G = 0, \quad \text{i.e. } g_p = -g_h \quad \text{on } S$$

Example 7.9

Obtain the Green's function for the Laplace's equation in the spherical domain.

Solution

Within the spherical domain with radius a, we consider

$$\nabla^2 G = \delta(\xi - x, \eta - y, \zeta - z)$$

which means

$$\nabla^2 g_h = \delta(\xi - x, \eta - y, \zeta - z)$$

and

$$\nabla^2 g_p = 0$$

For

$$r = [(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2]^{\frac{1}{2}} > 0$$

with (x, y, z) as the origin, we have

$$\nabla^2 g_h = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dg_h}{dr}\right) = 0$$

Integration then yields

$$g_h = A + \tfrac{B}{r} \quad \text{for } r > 0$$

Applying the condition (iii), we obtain

$$\lim_{\varepsilon\to 0}\iint_{S_\varepsilon} G_n dS = \lim_{\varepsilon\to 0}\iint_{S_\varepsilon} (g_h)_r dS = 1$$

from which we obtain $B = -\frac{1}{4\pi}$ and A is arbitrary. If we set $A = 0$ for convenience (this is the boundedness condition at infinity), we have

$$g_h = -\frac{1}{4\pi r}$$

To obtain the complete Green's functions, we need to find the solution for g_p. If we draw a three-dimensional diagram analogous to the two-dimensional as depicted in the last section, we will have a similar relation

$$r' = \frac{ar}{\rho} \tag{7.103}$$

where r' and ρ are measured in three-dimensional space.

Thus, we have

$$g_p = \frac{(\frac{a}{\rho})}{4\pi r'}$$

and hence

$$G = \frac{-1}{4\pi r} + \frac{\frac{a}{\rho}}{4\pi r'} \tag{7.104}$$

which is harmonic everywhere in r except at the source point, and is zero on the surface S. In terms of spherical coordinates:

$$\begin{aligned} \xi &= \tau \cos\psi \sin\alpha & x &= \rho \cos\phi \sin\theta \\ \eta &= \tau \sin\psi \sin\alpha & y &= \rho \sin\phi \sin\theta \\ \zeta &= \tau \cos\alpha & z &= \rho \cos\theta \end{aligned}$$

G can be written in the form

$$G = \frac{-1}{4\pi(\tau^2 + \rho^2 - 2\rho\tau\cos\gamma)^{\frac{1}{2}}} + \frac{1}{4\pi(\frac{\tau^2\rho^2}{a^2} + a^2 - 2\tau\rho\cos\gamma)^{\frac{1}{2}}} \tag{7.105}$$

where γ is the angle between r and r'.
Now differentiating G, we have

$$\left[\frac{\partial G}{\partial \tau}\right]_{\tau=a} = \frac{a^2 + \rho^2}{4\pi a(a^2 + \rho^2 - 2a\rho\cos\gamma)^{\frac{1}{2}}}$$

Thus, the Dirichlet problem for $h = 0$ is

$$u(x,y,z) = \frac{\alpha(a^2 - \rho^2)}{4\pi} \int_{\psi=0}^{2\pi} \int_{\alpha=0}^{\pi} \frac{f(\alpha,\psi)\sin\alpha d\alpha d\psi}{(a^2 + \rho^2 - 2a\rho\cos\gamma)^{\frac{1}{2}}} \tag{7.106}$$

where $\cos\gamma = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos(\psi - \phi)$. This integral is called the three-dimensional Poisson integral formula.

7.4.1 Green's function in 3D for physical problems

It is known, concerning the distribution of singularities in a flow field, that it is possible for certain bodies to be represented by combinations of sources and sinks and doublets. The combination of these singularities on the surrounding fluid may be used to represent complex body structures. We may think of a particular body as being composed of a continuous distribution of singularities: such singularities on the body surface will enable us to evaluate the velocity potential ϕ on that surface.

To deduce the appropriate formula, we consider two solutions of Laplace's equation in a volume V of fluid bounded by a closed surface S. Consider two potentials ϕ and ψ such that ϕ and ψ satisfy Laplace's equation in the following manner:

$$\nabla^2\phi = 0 \tag{7.107}$$

and

$$\nabla^2\psi = \delta(\vec{r} - \vec{r_0}) = \delta(x - \xi, y - \eta, z - \zeta), \tag{7.108}$$

where $\phi(x, y, z)$ is a regular potential of the problem and $\psi(x, y, z; \xi, \eta, \zeta)$ is the source potential which has a singularity at $\vec{r} = \vec{r_0}$. Here, δ is a Dirac delta function defined as

$$\delta(\vec{r} - \vec{r_0}) = \begin{cases} 0 & \text{when } \vec{r} \neq \vec{r_0} \\ \infty & \text{when } \vec{r} = \vec{r_0} \end{cases}$$

The source point $\vec{r_0} \equiv (\xi, \eta, \zeta)$ is situated at the body surface or inside the body. The point $\vec{r} = (x, y, z)$ can be regarded as the field point. There are three cases to investigate.

Case I

The field point (x, y, z) lies outside the body of surface area S and volume V (see Figure 7.5).

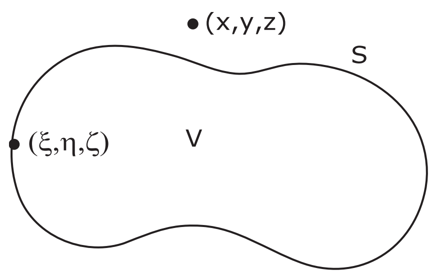

Figure 7.5: An arbitrary body surface.

By applying Green's theorem,

$$\begin{aligned}
\iint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] dS &= \iiint_V \vec{\nabla}(\phi\vec{\nabla}\psi) - \psi\vec{\nabla}\phi)dV \\
&= \iiint_V [\phi\nabla^2\psi + (\vec{\nabla}\phi)(\vec{\nabla}\psi) - \psi\nabla^2\phi - (\vec{\nabla}\psi)(\vec{\nabla}\phi)]dV \\
&= 0 \qquad (7.109)
\end{aligned}$$

Case II

The field point(x, y, z) lies inside the body of surface area S and volume V (see Figure 7.6). In this case

$$\begin{aligned}
\iint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\eta}{\partial n}\right] dS &= \iiint_V [\phi\nabla^2\psi - \psi\nabla^2\phi]dV \\
&= \iiint_V (\phi\nabla^2\psi)dV \\
&= \iint_V (\phi(x, y, z)\delta(x - \xi, y - \eta, z - \zeta))dV \\
&= \phi(\xi, \eta, \zeta) \qquad (7.110)
\end{aligned}$$

Now, changing the roles of (x, y, z) and (ξ, η, ζ), we obtain from equation (7.110)

$$\begin{aligned}
\phi(x, y, z) = \iint_S \bigg[&\phi(\xi, \eta, \zeta)\frac{\partial\psi}{\partial n}(x, y, z; \xi, \eta, \zeta) \\
&- \psi(x, y, z; \xi, \eta, \zeta)\frac{\partial\phi}{\partial n}(\xi, \eta, \zeta)\bigg] ds \qquad (7.111)
\end{aligned}$$

Referring to Figure 7.7 note that in (a) the point is interior to S, surrounded by a small spherical surface S_ε; (b) the field point is on the boundary surface S and S_ε is a hemisphere.

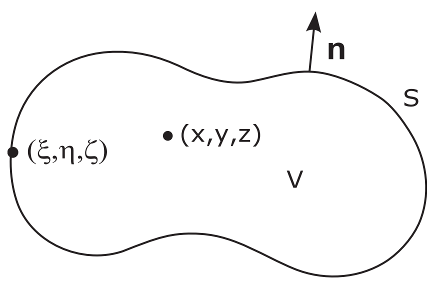

Figure 7.6: Surface integration for Green's theorem.

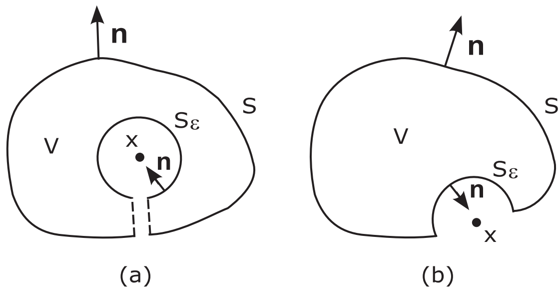

Figure 7.7: Singular points of surface integration for Green's theorem (inside and on the boundary).

Case III

The field point (x, y, z) lies on the body surface S within the volume V.

Referring to the work of Newman [11] on this subject, we may write:

$$\iint_S \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right] dS = \iiint_V \phi(x, y, z)\delta(x - \xi, y - \eta, z - \zeta)dV = \frac{1}{2}\phi(\xi, \eta, \zeta).$$

Changing the roles of (x, y, z) and (ξ, η, ζ), we obtain

$$\phi(x, y, z) = 2\iint_S [\phi(\xi, \eta, \zeta)\frac{\partial \psi}{\partial n}(x, y, z; \xi, \beta, \zeta) - \psi(x, y, z; \xi, \eta, \zeta)\frac{\partial \phi}{\partial n}(\xi, \eta, \zeta)]dS \tag{7.112}$$

Alternative method of deducing cases II and III

Consider the source potential

$$\psi = \tfrac{1}{4\pi r} = \tfrac{1}{4\pi}[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]^{-\frac{1}{2}}$$

where the field point (x, y, z) is well inside the body S. Then with reference to Figure 7.7, Green's divergence theorem yields

$$\iint_{S+S_\varepsilon} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS = 0,$$

where S_ε is a sphere of radius r. It follows that

$$\iint_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS = -\int_{S_\varepsilon} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS$$

$$\frac{1}{4\pi}\iint_S \left(\phi \frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial \phi}{\partial n}\right) dS = -\frac{1}{4\pi}\iint_{S_\varepsilon} \left[\phi \frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial \phi}{\partial n}\right] dS$$

Now using the concept of an outward normal, n, from the fluid,

$$\frac{\partial}{\partial n}\frac{1}{r} = -\frac{\partial}{\partial r}\frac{1}{r} = \frac{1}{r^2}$$

The right-hand side integral becomes

$$\begin{aligned}\frac{1}{4\pi}\iint_{S_\varepsilon}\left\{\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right\}dS &= \frac{1}{4\pi}\iint_{S_\varepsilon}\frac{\phi}{r^2}dS - \frac{1}{4\pi}\iint_{S_\varepsilon}\frac{1}{r}\left(\frac{\partial\phi}{\partial n}\right)dS \\ &= \frac{1}{4\pi}\cdot\frac{\phi(x,y,z)}{r^2}\cdot\iint_{S_\varepsilon}dS \\ &\quad -\frac{1}{4\pi}\cdot\frac{1}{r}\left(\frac{\partial\phi}{\partial n}\right)_{S_\varepsilon}\iint_{S_\varepsilon}dS \\ &= \frac{1}{4\pi}\frac{\phi(x,y,z)}{r^2}(4\pi r^2) - \frac{1}{4\pi}\frac{1}{r}\left(\frac{\partial\phi}{\partial n}\right)_{S_\varepsilon}(4\pi r^2) \\ &= \phi(x,y,z) - \left(\frac{\partial\phi}{\partial n}\right)_{S_\varepsilon} r.\end{aligned}$$

Thus, when $r \to 0$, $\left(\frac{\partial\phi}{\partial n}\right)_{S_\varepsilon} r \to 0$.

Combining these results, we have

$$\phi(x,y,z) = -\frac{1}{4\pi}\iint_S\left[\phi\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial\phi}{\partial n}\phi\right]dS.$$

This is valid when the field point (x,y,z) is inside S. The velocity potential $\phi(x,y,z)$ at a point inside a boundary surface S, can be written as (see Figure 7.8)

$$\phi(x,y,z) = \frac{1}{4\pi}\iint_S\frac{1}{r}\left(\frac{\partial\phi}{\partial n}\right)dS - \frac{1}{4\pi}\iint_S\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS$$

We know that the velocity potential due to a distribution of sources of strength m over a surface S is

$$\iint_S\frac{m}{r}dS,$$

and of a distribution of doublets of moments μ, the axes of which point inward along the normals to S, is

$$\iint_S\mu\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS.$$

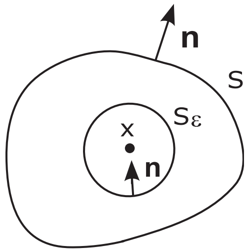

Figure 7.8: Singular point inside the surface integration.

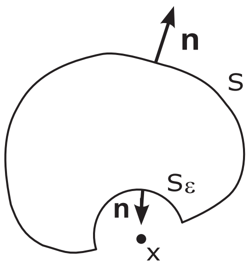

Figure 7.9: Singular point on the boundary of the surface.

Thus, the velocity potential ϕ at a point (x, y, z), as given above, is the same as if the motion in the region bounded by the surface S due to a distribution over S of simple sources, of density $(\frac{1}{4\pi})(\frac{\partial\phi}{\partial n})$ per unit area, together with a distribution of doublets, with axes pointing inwards along the normals to the surfaces, of density $(\frac{\phi}{4\pi})$ per unit area. When the field point (x, y, z) is on S, as shown in Figure 7.9, the surface S_ε is a hemisphere of surface area $2\pi r^2$.

When $\varepsilon \to 0$, $\iint_{S_\varepsilon} \frac{\phi}{r^2} dS = 2\pi\phi(x, y, z)$, then

$$\phi(x, y, z) = -\frac{1}{2\pi} \iint \left[\phi \frac{\partial}{\partial n} \frac{1}{r} - \frac{1}{r} \frac{\partial}{\partial n} \phi \right] ds$$

Summary

Thus, summarizing all these results, we obtain

(i) When the point (x, y, z) lies outside S, then

$$\iint_S \left(\phi \frac{\partial}{\partial n} \frac{1}{r} - \frac{1}{r} \frac{\partial \phi}{\partial n} \right) dS = 0.$$

(ii) When the point (x, y, z) lies inside S, then

$$\phi(x, y, z) = -\frac{1}{4\pi} \iint_S \left(\phi \frac{\partial}{\partial n} \frac{1}{r} - \frac{1}{r} \frac{\partial \phi}{\partial n} \right) dS.$$

(iii) When the point (x, y, z) lies on the boundary S, then

$$\phi(x, y, z) = -\frac{1}{2\pi}\iint_S \left(\phi\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial\phi}{\partial n}\right)dS,$$

where ϕ is known as the velocity potential of the problem.

Note that the last equation is frequently used for obtaining the velocity potential due to the motion of a ship's hull. The normal velocity, $\frac{\partial\phi}{\partial n}$, is known on the body, so that the last equation is an integral equation for determining the unknown potential; this may be done by numerical integration.

In many practical problems, however, the body may move in a fluid bounded by other boundaries, such as the free surface, the fluid bottom, or possibly lateral boundaries such as canal walls.

In this context, we use Green's function

$$G(x, y, z; \xi, \eta, \zeta) = \tfrac{1}{r} + H(x, y, z; \xi, \eta, \zeta), \tag{7.113}$$

where

$$\nabla^2 H = \delta(x - \xi, y - \eta, z - \zeta) \tag{7.114}$$

Green's function, defined above, can be stated as

$$\iint_S \left(\phi\frac{\partial G}{\partial n} - G\frac{\partial\phi}{\partial n}\right)dS = \begin{cases} 0 \\ -2\pi\phi(x, y, z) \\ -4\pi\phi(x, y, z) \end{cases} \tag{7.115}$$

for (x, y, z) outside, on, or inside the closed surface S, respectively.

If a function H can be found with the property that $\frac{\partial\phi}{\partial n} = 0$ on the boundary surfaces of the fluid, then equation (7.106) may be rewritten as

$$\iint_S G(x, y, z; \xi, \eta, \zeta)\frac{\partial\phi}{\partial n}(\xi, \eta, \zeta)dS = \begin{cases} 0 \\ 2\pi\phi(x, y, z) \\ 4\pi\phi(x, y, z) \end{cases} \tag{7.116}$$

for (x, y, z) outside, on, or inside the closed surface S, respectively. Here, $\frac{\partial\phi}{\partial n}(\xi, \eta, \zeta) = Q(\xi, \eta, \zeta)$ is defined to be the unknown source density (strength) and has to be evaluated by numerical methods from the above integral. Once the source density is known, the field potential $\phi(x, y, z)$ can be easily obtained.

7.4.2 Application: hydrodynamic pressure forces

One of the main purposes for studying the fluid motion past a body is to predict the wave forces and moments acting on the body due to the hydrodynamic pressure of the fluid.

The wave forces and moments can be calculated using the following formulae:

$$\vec{F} = \iint_{S_B} (P\vec{n}dS) \tag{7.117}$$

$$\vec{M} = \iint_{S_B} P(\vec{r}X\vec{n})dS, \tag{7.118}$$

where S_B is the body of the structure and $\vec{n}$ is the unit normal vector, which is positive when pointing out of the fluid volume.

From Bernoulli's equation, we know that

$$P = -\rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz\right], \tag{7.119}$$

where $\frac{\partial\phi}{\partial t}$ is the transient pressure, $\frac{1}{2}(\nabla\phi)^2$ the dynamic pressure, and gz the static pressure. Then using equation (7.110) with equations (7.108) and (7.109), we obtain

$$\vec{F} = -\rho\iint_{S_B}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz\right]\vec{n}dS \tag{7.120}$$

$$\vec{M} = -\rho\iint_{S_B}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz\right](\vec{r}X\vec{n})dS \tag{7.121}$$

In the following section, we shall deduce Green's function which is needed in the evaluation of the velocity potential from the integral equation (7.107). The solution is mainly due to Wehausen and Laitone [17].

7.4.3 Derivation of Green's function

We will obtain Green's function solution (singular solution) for the cases of infinite depth and of finite depth. Mathematical solutions will be first obtained for the infinite depth case, and then extended to the case of finite depth.

Case I: Infinite depth

Consider Green's function of the form

$$G(x, y, z; \xi, \eta, \zeta, t) = Re\{\vec{g}(x, y, z; \xi, \eta, \zeta)e^{-i\sigma t}\}, \tag{7.122}$$

where Re stands for the real part of the complex function, and $\vec{g}$ is a complex function which can be defined as

$$\vec{g}(x, y, z; \xi, \eta, \zeta) = g_1(x, y, z; \xi, \eta, \zeta) + ig_2(x, y, z; \xi, \eta, \zeta) \tag{7.123}$$

Here, g_1 and g_2 are real functions. Thus,

$$G(x, y, z; \xi, \eta, \zeta, t) = g_1 \cos\sigma t + g_2 \sin\sigma t \tag{7.124}$$

G satisfies Laplace's equation except at the point (ξ, η, ζ), where

$$\nabla^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} = \delta(x, y, z; \xi, \eta, \zeta) \tag{7.125}$$

From the above, the equations and conditions to be satisfied by g_1 and g_2 are as follows:

$$\nabla^2 g_1 = \delta(x, y, z; \xi, \eta, \zeta), \quad \nabla^2 g_2 = \delta(x, y, z; \xi, \eta, \zeta) \tag{7.126}$$

The linear surface boundary condition given by

$$\frac{\partial^2 G}{\partial t^2} + g\frac{\partial G}{\partial z} = 0, \quad \text{for } z \leq 0, \tag{7.127}$$

yields to

$$\frac{\partial g_1}{\partial z} - \frac{\sigma^2}{g} g_1 = 0, \quad \frac{\partial g_2}{\partial z} - \frac{\sigma^2}{g} g_2 = 0 \quad \text{at } z = 0 \tag{7.128}$$

The bottom boundary conditions for the infinite depth case are

$$\lim_{z \to -\infty} \vec{\nabla} g_1 = 0, \qquad \lim_{z \to -\infty} \vec{\nabla} g_2 = 0. \tag{7.129}$$

The radiation condition can be stated as

$$\lim_{R \to \infty} \sqrt{R} \left(\frac{\partial \vec{g}}{\partial R} - ik\vec{g} \right) = 0, \tag{7.130}$$

which yields

$$\lim_{R \to \infty} \sqrt{R} \left(\frac{\partial g_1}{\partial R} + kg_2 \right) = 0, \quad \lim_{R \to \infty} \sqrt{R} \left(\frac{\partial g_2}{\partial R} - kg_1 \right) = 0. \tag{7.131}$$

Here, k is the wavenumber and R is the radial distance in the $x-y$-plane and is given by $R^2 = (x-\xi)^2 + (y-\eta)^2$.

We shall now assume a solution of G, as given in equation (7.115) to be in the following form:

$$\begin{aligned} G &= \left(\tfrac{1}{r} + g_0\right) \cos \sigma t + g_2 \sin \sigma t \\ &= \tfrac{1}{r} \cos \sigma t + g_0 \cos \sigma t + g_2 \sin \sigma t, \end{aligned} \tag{7.132}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$, and $g_1 = \frac{1}{r} + g - 0$.

Denoting the double Fourier transform in x and y of $\vec{g}$ by g^* as

$$\vec{g}(x,y,z;\xi,\eta,\zeta) = \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} g^*(k,\theta,z;\xi,\eta,\zeta)e^{ik(x\cos\theta+y\sin\theta)}d\theta dk, \quad (7.133)$$

we then can write

$$g_0(x,y,z;\xi,\eta,\zeta) = \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} g_0^*(k,\theta,z;\xi,\eta,\zeta)e^{ik(x\cos\theta+y\sin\theta)}d\theta dk, \quad (7.134)$$

and

$$g_2(x,y,z;\xi,\eta,\zeta) = \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} g_2^*(k,\theta,z;\xi,\eta,\zeta)e^{ik(x\cos\theta+y\sin\theta)}d\theta dk. \quad (7.135)$$

Note that functions g_0 and g_2 happen to be regular functions of x, y, z, and satisfy Laplace's equation.

Applying the double Fourier transform in x and y of g_0 yields

$$\frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} \left(\frac{\partial^2 g_0^*}{\partial z^2} - k^2 g_0^*\right) e^{ik(x\cos\theta+y\sin\theta)}d\theta dk = 0,$$

and consequently, we obtain

$$\frac{\partial^2 g_0^*}{\partial z^2} - k^2 g_0^* = 0. \quad (7.136)$$

Solving equation (7.127) we get

$$g_0^* = A(k,\theta)e^{kz} + B(k,\theta)e^{-kz}. \quad (7.137)$$

Since g_0^* must be bounded as $z \to -\infty$, then $B = 0$, and the solution becomes

$$g_0^* = A(k,\theta)e^{kz} \quad (7.138)$$

We know that

$$\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} e^{-k|z|}e^{ik(x\cos\theta+y\sin\theta)}d\theta dk. \quad (7.139)$$

Extending this result, we obtain

$$\begin{aligned}
\frac{1}{r} &= \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} \\
&= \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} e^{-k|z-\zeta|} e^{ik((x-\xi)\cos\theta+(y-\eta)\sin\theta)} d\theta dk \\
&= \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} e^{-k|z-\zeta|} e^{-ik(\xi\cos\phi+\eta\sin\theta)} \\
&\quad \times e^{ik(x\cos\theta+y\sin\theta)} d\theta dk \\
&= \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} \left(\frac{1}{r}\right)^* e^{ik(x\cos\theta+y\sin\theta)} d\theta dk, \qquad (7.140)
\end{aligned}$$

where

$$\left(\tfrac{1}{r}\right)^* = e^{-k|z-\zeta|} e^{-ik(\xi\cos\theta+\eta\sin\theta)}. \qquad (7.141)$$

Taking the double Fourier transform of the surface boundary condition, equation (7.119) yields

$$\left\{\frac{\partial g_0^*}{\partial z} - k\left(\frac{1}{r}\right)^*\right\} - \frac{\sigma^2}{g}\left(g_0^* + \left(\frac{1}{r}\right)^*\right) = 0 \quad \text{at z} = 0 \qquad (7.142)$$

Rearranging the terms yields

$$\frac{\partial g_0^*}{\partial z} - \frac{\sigma^2}{g} g_0^* = \left(k + \frac{\sigma^2}{g}\right)\left(\frac{1}{r}\right)^* \quad \text{at z} = 0 \qquad (7.143)$$

From equation (7.134) and the boundary condition, i.e. equation (7.129), we have

$$A(k,\theta) = \frac{k+\sigma^2/g}{k-\sigma^2/g} e^{k\zeta} e^{-ik(\xi\cos\theta+\eta\sin\theta)} \qquad (7.144)$$

Therefore,

$$g_0^* = \frac{k+\sigma^2/g}{k-\sigma^2/g} e^{k(z+\zeta)} e^{-ik(\xi\cos\theta+\eta\sin\theta)} \qquad (7.145)$$

Inverting this function gives

$$g_0 = \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} \frac{k+\sigma^2/g}{k-\sigma^2/g} e^{k(z+\zeta)} e^{ik((x-\xi)\cos\theta+(y-\eta)\sin\theta)} d\theta dk. \qquad (7.146)$$

If we define $\frac{\sigma^2}{g} = \upsilon$, then the potential g_1 may be written as

$$\begin{aligned} g_1(x,y,z) &= \tfrac{1}{r} + g_0(x,y,z) \\ &= \frac{1}{r} + \frac{1}{2\pi} P.V. \int_0^\infty \int_{-\pi}^{\pi} \frac{k+\upsilon}{k-\upsilon} e^{k(z+\zeta)} \\ &\quad \times e^{ik((x-\xi)\cos\theta + (y-\eta)\sin\theta)} d\theta dk \end{aligned} \tag{7.147}$$

This may be written as

$$g_1(x,y,z) = \frac{1}{r} + \frac{1}{r_1} + \frac{\upsilon}{\pi} P.V. \int_0^\infty \int_{-\pi}^{\pi} \frac{e^{k(z+\zeta)}}{k-\upsilon} e^{ik((x-\xi)\cos\theta + (y-\eta)\sin\theta)} d\theta dk. \tag{7.148}$$

where

$$r_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}$$

Here, $P.V.$ stands for the Cauchy principal value. Note that g_1 satisfies all the given boundary conditions except the radiation condition. To satisfy the radiation condition, we need the asymptotic expansion of g_1. The solution, i.e. equation (7.148) may be written in the form

$$g_1(x,y,z) = \frac{1}{r} + \frac{1}{r_1} + \frac{4\upsilon}{\pi} P.V. \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{e^{k(z+\zeta)}}{k-\upsilon} \cos(kR\cos\theta) d\theta dk, \tag{7.149}$$

where $R = \sqrt{(x-\xi)^2 + (y-\eta)^2}$. But $\cos\theta = \lambda$ and $-\sin\theta d\theta = d\lambda$, which on substitution into equation (7.149) yields

$$g_1(x,y,z) = \frac{1}{r} + \frac{1}{r_1} + \frac{4\upsilon}{\pi} P.V. \int_0^\infty \int_0^1 \frac{e^{k(z+\zeta)}}{k-\upsilon} \frac{\cos(kR\lambda)}{\sqrt{1-\lambda^2}} d\lambda dk. \tag{7.150}$$

We know that

$$\frac{2}{\pi} \int_0^1 \frac{\cos(kR\lambda)}{\sqrt{1-\lambda^2}} d\lambda = J_0(kR),$$

and hence

$$\begin{aligned} g_1(x,y,z) &= \frac{1}{r} + \frac{1}{r_1} + 2\lambda P.V. \int_0^\infty \frac{e^{k(z+\zeta)}}{k-\upsilon} J_0(kR) dk \\ &= \frac{1}{r} + P.V. \int_0^\infty \frac{k+\upsilon}{k-\upsilon} e^{k(z+\zeta)} J_0(kR) dk. \end{aligned} \tag{7.151}$$

To determine the asymptotic form of g_1, when R goes to infinity, we will use the following Fourier integrals:

$$\left.\begin{aligned} \int_a^\infty f(x)\frac{\sin R(x-x_0)}{x-x_0}dx &= \pi f(x_0)+O\left(\frac{1}{R}\right) \\ P.V.\int_a^\infty f(x)\frac{\cos R(x-x_0)}{x-x_0}dx &= O\left(\frac{1}{R}\right) \end{aligned}\right\} \tag{7.152}$$

for $a \le x_0 \le \infty$. We know that when $R \to \infty$

$$\frac{1}{r}=O\left(\frac{1}{R}\right), \quad \frac{1}{r_1}=O\left(\frac{1}{R}\right)$$

and

$$g_1(x,y,z)=\frac{4\upsilon}{\pi}P.V.\int_0^\infty\int_0^1\frac{1}{\sqrt{1-\lambda^2}}\cdot\frac{e^{k(z+\zeta)}}{k-\upsilon}.$$

$$[\cos(R\lambda\upsilon)\cos R\lambda(k-\upsilon)-\sin(R\lambda\upsilon)\sin R\lambda(k-\upsilon)]d\lambda dk+O\left(\tfrac{1}{R}\right).$$

Using the formulas (7.152), this equation can be reduced to

$$g_1(x,y,z)=-4\lambda e^{\upsilon(z+\zeta)}\int_0^1\frac{\sin(r\lambda\upsilon)}{\sqrt{1-\lambda^2}}d\lambda+O\left(\frac{1}{R}\right),$$

which subsequently (see Rahman [13]) can be reduced to the following:

$$g_1(x,y,z)=-2\pi\upsilon e^{\upsilon(z+\zeta)}\sqrt{\frac{2}{\pi R\upsilon}}\sin\left(R\upsilon-\frac{\pi}{4}\right)+O\left(\frac{1}{R}\right). \tag{7.153}$$

From the radiation conditions, we can at once predict the asymptotic form of $g_2(x,y,z)$, which is

$$g_2(x,y,z)=2\pi\upsilon e^{\upsilon(z+\zeta)}\sqrt{\frac{2}{\pi R\upsilon}}\cos\left(R\upsilon-\frac{\pi}{4}\right)+O\left(\frac{1}{R}\right). \tag{7.154}$$

Thus, the asymptotic form of $G=g_1\cos\sigma t+g_2\sin\sigma t$ is

$$-2\pi\upsilon e^{\upsilon(z+\zeta)}\sqrt{\frac{2}{\pi R\upsilon}}\sin\left(R\upsilon-\sigma t-\frac{\pi}{4}\right)+O\left(\frac{1}{R}\right). \tag{7.155}$$

It can be easily verified that g_1 has the same asymptotic behaviour as

$$-2\pi\upsilon e^{(z+\zeta)}Y_0(R\upsilon),$$

and therefore the function $g_2(x,y,z)$ will be

$$g_2=2\pi\upsilon e^{\upsilon(z+\zeta)}J_0(R\upsilon) \tag{7.156}$$

which satisfies all the required conditions. Here, $J_0(R\upsilon)$ and $Y_0(R\upsilon)$ are the Bessel functions of the first and second kind, respectively.

Combining all these results, the final real solution form of G is

$$G(x,y,z;\xi,\eta,\zeta,t) = \left[\frac{1}{r} + P.V.\int_0^\infty \frac{k+\upsilon}{k-\upsilon} e^{k(z+\zeta)} J_0(kR)dk\right]\cos\sigma t$$

$$+ 2\pi\upsilon e^{\upsilon(z+\zeta)} J_0(\upsilon R)\sin\sigma t. \tag{7.157}$$

The complex form is

$$\begin{aligned} \vec{g} &= g_1 + ig_2 \\ &= \left[\frac{1}{r} + P.V.\int_0^\infty \frac{k+\upsilon}{k-\upsilon} e^{k(z+\zeta)} J_0(kR)dk\right] \\ &\quad + i2\pi\upsilon e^{\upsilon(z+\zeta)} J_0(\upsilon r) \end{aligned} \tag{7.158}$$

Case II: Finite depth case

Consider Green's function of the form

$$G = Re\{\vec{g}e^{-i\sigma t}\} = g_1\cos\sigma t + g_2\sin\sigma t, \tag{7.159}$$

where

$$g_1 = \tfrac{1}{r} + \tfrac{1}{r_2} + g_0(x,y,z)$$

and

$$r_2^2 = (x-\xi)^2 + (y-\eta)^2 + (z+2h+\zeta)^2$$

The function g_1 satisfies Laplace's equation

$$\nabla^2 g_1 = 0,$$

and therefore

$$\nabla^2\left(\tfrac{1}{r} + \tfrac{1}{r_2} + g_0\right) = 0$$

We can easily verify that

$$\nabla^2\left(\tfrac{1}{r}\right) = 0 \quad \text{and} \quad \nabla^2\left(\tfrac{1}{r_2}\right) = 0$$

Therefore,

$$\nabla^2 g_0 = 0 \tag{7.160}$$

Applying the double Fourier transform

$$g_0(x,y,z) = \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} g_0^*(k,\theta,z)e^{ik(x\cos\theta + y\sin\theta)}d\theta dk$$

to Laplace's equation (7.160), we obtain

$$\frac{\partial^2 g_0^*}{\partial z^2} - kg_0^* = 0, \tag{7.161}$$

the solution of which can be written as

$$g_0^* = Ae^{kz} + Be^{-kz}. \tag{7.162}$$

The constants A and B must be evaluated using the bottom boundary conditions and the free surface boundary condition.

The bottom boundary condition is given by

$$\frac{\partial g_1}{\partial z}(x,y,z=-h) = 0 \tag{7.163}$$

Thus,

$$\frac{\partial}{\partial z}\left(\frac{1}{r} + \frac{1}{r_2}\right) + \left(\frac{\partial g_0}{\partial z}\right) = 0 \quad \text{at } z = -h.$$

It can be easily verified that

$$\frac{\partial}{\partial z}\left(\frac{1}{r} + \frac{1}{r_2}\right) = 0 \quad \text{at } z = -h,$$

if we choose

$$r_2^2 = (x-\xi)^2 + (y-\eta)^2 + (z+2h+\zeta)^2$$

Thus, the bottom boundary condition to be satisfied by g_0 is

$$\frac{\partial g_0}{\partial z} = 0 \quad \text{at } z = -h \tag{7.164}$$

The double Fourier transform of equation (7.164) yields

$$\frac{\partial g_0^*}{\partial z} = 0 \quad \text{at } z = -h \tag{7.165}$$

Using this condition in the solution (7.162), we obtain $B = Ae^{-2kh}$ and hence

$$g_0^* = Ae^{-kh}[e^{k(z+h)} + e^{-k(z+h)}] = C\cosh k(z+h) \tag{7.166}$$

where $C = 2Ae^{-kh}$ a redefined constant. To evaluate C, we have to satisfy the free surface condition

$$\frac{\partial g_1}{\partial z} - \upsilon g_1 = 0 \quad \text{at } z = 0,$$

that is

$$\frac{\partial}{\partial z}\left(\frac{1}{r} + \frac{1}{r_2} + g_0\right) - \upsilon\left(\frac{1}{r} + \frac{1}{r_2} + g_0\right) = 0 \quad \text{at } z = 0. \tag{7.167}$$

Taking the double Fourier transform of equation (7.167), we obtain

$$\frac{\partial g_0^*}{\partial z} - \upsilon g_0^* = (k+\upsilon)\left(\frac{1}{r} + \frac{1}{r_2}\right)^* \quad \text{at } z = 0. \tag{7.168}$$

Using this condition in equation (7.166), we obtain

$$C = \frac{k+\upsilon}{k\sinh kh - \upsilon\cosh kh}\left(\frac{1}{r} + \frac{1}{r_2}\right)^* \quad z = 0. \tag{7.169}$$

We know that

$$\left(\tfrac{1}{r}\right)^* = e^{-k|z-\zeta|}e^{-ik(\xi\cos\theta+\eta\sin\theta)}$$

and

$$\left(\frac{1}{r_2}\right)^* = e^{-k|z+2h+\zeta|}e^{-ik(\xi\cos\theta+\eta\sin\theta)}$$

Hence, at $z = 0$

$$\left(\tfrac{1}{r}\right)^* = e^{k\xi}e^{-ik(\xi\cos\theta+\eta\sin\theta)}$$

and

$$\left(\frac{1}{r_2}\right)^* = e^{-k\zeta}e^{-2kh}e^{-ik(\xi\cos\theta\,\eta\sin\theta)}.$$

Therefore,

$$\begin{aligned}\left(\tfrac{1}{r}\right)^* + \left(\frac{1}{r_2}\right)^* &= e^{-kh}(e^{k(h+\zeta)} + e^{-k(h+\zeta)})e^{-ik(\xi\cos\theta+\eta\sin\theta)} \\ &= 2e^{-kh}\cosh k(h+\zeta)e^{-ik(\xi\cos\theta+\eta\sin\theta)}.\end{aligned}$$

Substituting this into equation (7.166) yields

$$C = \frac{2e^{-kh}(k+\upsilon)\cosh k(h+\zeta)}{k\sinh kh - \upsilon\cosh kh} e^{-ik(\xi\cos\theta + \eta\sin\theta)}, \tag{7.170}$$

and consequently

$$g_0^* = \frac{2(k+\upsilon)e^{-kh}\cosh k(h+\zeta)\cosh k(z+h)}{k\sinh kh - \upsilon\cosh kh} e^{-ik(\xi\cos\theta+\eta\sin\theta)} \tag{7.171}$$

Inverting this expression, we can write the g_1 solution as

$$g_1 = \frac{1}{r} + \frac{1}{r_2} + \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} \frac{2(k+\upsilon)e^{-kh}\cosh k(h+\zeta)}{k\sinh kh - \upsilon\cosh kh} \times \cosh k(z+h) e^{-ik((x-\xi)\cos\theta+(y-\eta)\sin\theta)} d\theta dk \tag{7.172}$$

which can subsequently be written as

$$g_1 = \frac{1}{r} + \frac{1}{r_2} + \int_0^\infty \frac{2(k+\upsilon)e^{-kh}\cosh k(h+\zeta)\cosh k(z+\zeta)}{k\sinh kh - \upsilon\cosh kh} J_0(kR)dk$$

To satisfy the radiation condition we must first obtain the asymptotic expansion ($R \to \infty$) for g_1.

Consider the integral

$$g_0 = \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} \frac{2(k+\upsilon)e^{-kh}\cosh k(h+\zeta)\cosh k(z+h)}{k\sinh kh - \upsilon\cosh kh} \times e^{ik((x-\xi)\cos\theta+(y-\eta)\sin\theta)} d\theta dk$$

Since $x-\xi = R\cos\delta$ and $y-\eta = R\sin\delta$, then

$$\begin{aligned} R &= (x-\xi)\cos\delta + (y-\eta)\sin\delta \\ &= \sqrt{(x-\xi)^2 + (y-\eta)^2}. \end{aligned}$$

Also

$$e^{ik(x-\xi)\cos\theta+(y-\eta)\sin\theta)} = e^{ikR\cos(\theta-\delta)}$$

$$\begin{aligned} \text{and hence} \int_{-\pi}^{\pi} e^{ikR\cos(\theta-\delta)} d\theta &= 4\int_0^{\frac{\pi}{2}} \cos(kR\cos\theta) d\theta \\ &= 4\int_0^1 \frac{\cos(kr\lambda)}{\sqrt{1-\lambda^2}} d\lambda \\ &= 2\pi J_0(kR), \end{aligned}$$

where

$$J_0(kR) = \frac{2}{\pi}\int_0^1 \frac{\cos(kR\lambda)}{\sqrt{1-\lambda^2}}d\lambda$$

Thus,

$$g_0 = \frac{2}{\pi}\int_0^1 \left[\int_0^\infty \frac{2(k+\upsilon)e^{-kh}\cosh k(z+\zeta)\cosh k(h+\zeta)}{\frac{(k\sinh kh-\upsilon\cosh kh)}{k-m_0}} \right.$$
$$\left. \times \frac{1}{\sqrt{1-\lambda^2}}\left(\frac{\cosh(kR\lambda)}{k-m_0}\right)dk\right]d\lambda,$$

where

$$\cos(kR\lambda) = \cos(\lambda R m_0)\cos\lambda R(k-m_0) - \sin(\lambda R m_0)\sin\lambda R(k-m_0).$$

Using the Fourier integrals in the form, i.e. equation (7.152) gives

$$g_0 = \frac{-4(m_0+\upsilon)e^{-m_0h}\cosh m_0(z+\zeta)\cosh m_0(h+\zeta)}{\lim_{R\to m_0}\frac{(k\sinh kh-\upsilon\cosh kh)}{k-m_0}}$$
$$\times \int_0^1 \frac{\sin(R\lambda m_0)}{\sqrt{1-\lambda^2}}d\lambda + O\left(\frac{1}{R}\right),$$

where m_0 is a root of $m_0h\tanh m_0h = \upsilon h$. Now,

$$\lim_{k\to m_0}\frac{k\sinh kh-\upsilon\cosh kh}{k-m_0}$$
$$= \lim_{k\to m_0}[kh\cosh kh + \sinh kh - \upsilon h\sinh kh]$$
$$= m_0h\cosh m_0h + \sinh m_0h - \upsilon h\sinh m_0h$$
$$= \frac{\upsilon h\cosh^2 m_0h + \sinh^2 m_0h - \upsilon h\sinh^2 m_0h}{\sinh m_0h}$$
$$= \frac{\upsilon h + \sinh^2 m_0h}{\sinh m_0h}$$

Therefore, the asymptotic behaviour of $g_0(x,y,z)$ as $R\to\infty$ is

$$g_0 = -\frac{4(m_0+\upsilon)\cosh m_0(z+\zeta)\cosh m_0(h+\zeta)\sinh m_0h}{\upsilon h+\sinh^2 m_0h}$$
$$\times \int_0^1 \frac{\sin(R\lambda m_0)}{\sqrt{1-\lambda^2}}d\lambda + O\left(\frac{1}{R}\right)$$

However, the asymptotic behaviour of the integral term is given by

$$\int_0^1 \frac{\sinh(\lambda R m_0)}{\sqrt{1-\lambda^2}} d\lambda = \frac{\pi}{2}\sqrt{\frac{2}{\pi R m_0}} \sin\left(Rm_0 - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right) \tag{7.173}$$

Thus,

$$g_0 = -\frac{2\pi(m_0+\upsilon)e^{-m_0 h}\sinh m_0 h \cosh m_0(z+\zeta)\cosh m_0(h+\zeta)}{\upsilon h + \sinh^2 m_0 h} \times \sqrt{\frac{2}{\pi m_0 R}} \sin\left(Rm_0 - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right) \tag{7.174}$$

Now, using the radiation condition

$$\sqrt{R}\left[\frac{\partial g_0}{\partial R} + m_0 g_2\right] = 0, \qquad \text{as } R \to \infty, \tag{7.175}$$

we obtain

$$g_2 = \frac{2\pi(m_0+\upsilon)e^{-m_0 h}\sinh m_0 h \cosh m_0(z+\zeta)\cosh m_0(h+\zeta)}{\upsilon h + \sinh^2 m_0 h} \times \sqrt{\frac{1}{\pi m_0 R}} \cos\left(Rm_0 - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right).$$

It can be easily verified that the function

$$g_2 = \frac{2\pi(m_0 h)e^{-m_0 h}\sinh m_0 h \cosh m_0(z+\zeta)\cosh m_0(h+\zeta)}{\upsilon h + \sinh^2 m_0 h} J_0(m_0 R) \tag{7.176}$$

will satisfy all the required boundary conditions including Laplace's equation.

Thus, combining these results, the final form of the velocity potential G may be expressed as

$$\begin{aligned} G(x,y,z;\xi,\eta,\zeta,t) = \Bigg[&\frac{1}{r} + \frac{1}{r_2} + P.V.\int_0^\infty \frac{2(k+\upsilon)e^{-kh}\cosh k(h+\zeta)}{k\sinh kh - \upsilon\cosh kh} \\ &\times \cosh k(z+h) J_0(kR)dk\Bigg] \cos\sigma t \\ &+ \frac{2\pi(m_0+\upsilon)e^{-m_0 h}\sinh m_0 h \cosh m_0(h+\zeta)}{\upsilon h + \sinh^2 m_0 h} \\ &\times \cosh m_0(z+h) J_0(m_0 R)\sin\sigma t, \end{aligned} \tag{7.177}$$

where

$$m_0 \tanh m_0 h = \upsilon = \frac{\sigma^2}{g}.$$

The above results are due to Wehausen and Laitone [17]. John [7] has derived the following Green's function in terms of the infinite series:

$$\begin{aligned}
G(x,y,z;\xi,\eta,\zeta,t) = 2\pi \frac{\upsilon^2 - m_0^2}{hm_0^2 - h\upsilon^2 + \upsilon} \cosh m_0(z+h) \\
\times \cosh m_0(h+\zeta)[Y_0(m_0R)\cos\sigma t - J_0(m_0R)\sin\sigma t] \\
+4\sum_{k=1}^{\infty} \frac{m_k^2 + \upsilon^2}{hm_k^2 + h\upsilon^2 - \upsilon} \cos m_k(z+h)\cosh m_k(h+\zeta) \\
\times K_0(m_kR)\cos\sigma t,
\end{aligned} \tag{7.178}$$

where $m_k, k > 0$ are the positive real roots of $m\tan mh + \upsilon = 0$, and $K_0(m_kR)$ is the modified Bessel function of second kind of zeroth-order.

7.5 Numerical formulation

Green's function provides an elegant mathematical tool to obtain the wave loadings on arbitrarily shaped structures. We know that the total complex velocity potential $\phi(x,y,z,t)$, in diffraction theory, may be expressed as the sum of an incident wave potential ϕ_I and a scattered potential ϕ_S in the following manner:

$$\Phi(x,y,z,t) = Re\{\phi(x,y,z)e^{-i\sigma t}\}, \tag{7.179}$$

where Φ is the real velocity potential, $\phi(x,y,z)$ the complex potential, and σ the oscillatory frequency of the harmonic motion of the structure. Thus,

$$\phi = \phi_I + \phi_S \tag{7.180}$$

The complex potentials should satisfy the following conditions:

$$\left.\begin{array}{ll}
\nabla^2\phi = 0 & \text{in the fluid interior} \\
\dfrac{\partial\phi}{\partial z} - \upsilon\phi = 0 & \text{on the free surface} \\
\dfrac{\partial\phi_I}{\partial n} = -\dfrac{\partial\phi_S}{\partial n} & \text{on the fixed structures} \\
\dfrac{\partial\phi_I}{\partial n} = \dfrac{\partial\phi_S}{\partial n} = 0 & \text{on the sea bottom} \\
\lim_{R\to\infty}\sqrt{R}\left(\dfrac{\partial\phi_S}{\partial R} - ik\phi_S\right) = 0 & \text{at far field}
\end{array}\right\} \tag{7.181}$$

R is the horizontal polar radius.

Hydrodynamic pressure from the linearized Bernoulli equation is

$$P = -\rho\left(\frac{\partial\phi}{\partial t}\right) = \rho Re[i\sigma\phi] = \rho Re[i\sigma(\phi_I + \phi_S)] \tag{7.182}$$

The force and moment acting on the body may be calculated from the formulae

$$\vec{F} = -\int\int_S P\vec{n}dS \tag{7.183}$$

$$\vec{M} = -\int\int_S P(\vec{r}X\vec{n})dS \tag{7.184}$$

In evaluating the wave loading on the submerged structures, we first need to find the scattered potentials, ϕ_S, which may be represented as being due to a continuous distribution of point wave sources over the immersed body surface. As indicated earlier, a solution for ϕ_S at the point (x, y, z) may be expressed as:

$$\phi_S(x,y,z) = \frac{1}{4\pi}\int\int_S Q(\xi,\eta,\zeta)G(x,y,z;\xi,\eta,\zeta)dS \tag{7.185}$$

The integral is over all points (ξ, η, ζ) lying on the surface of the structure, $Q(\xi, \eta, \zeta)$ is the source strength distribution function, and dS is the differential area on the immersed body surface. Here, $G(x, y, z; \xi, \eta, \zeta)$ is a complex Green's function of a wave source of unit strength located at the point (ξ, η, ζ).

Such a Green's function, originally developed by John [7] and subsequently extended by Wehausen and Laitone [17], was illustrated in the previous section.

The complex form of G may be expressed either in integral form or in infinite series form.

The integral form of G is as follows:

$$\begin{aligned} G(x,y,z;\xi,\eta,\zeta) &= \frac{1}{r} + \frac{1}{r_2} \\ &+ P.V. \int_0^\infty \frac{2(k+\upsilon)e^{-kh}\cosh(k(\zeta+h))\cosh(k(z+h))}{k\sinh kh - \upsilon\cosh kh} J_0(kR)dk - 2\pi i \\ &\times \frac{(m_0+\upsilon)e^{-m_0 h}\sinh m_0 h\cosh m_0(h+\zeta)\cosh m_0(z+h)}{\upsilon h + \sinh^2 m_0 h} J_0(m_0 R) \end{aligned} \tag{7.186}$$

Since

$$\frac{e^{-m_0 h}\sinh m_0 h}{\upsilon h + \sinh^2 m_0 h} = \frac{m_0 - \upsilon}{(m_0^2 - \upsilon^2)h + \upsilon}, \tag{7.187}$$

then equation (7.186) may be expressed as

$$\begin{aligned}
G(x,y,z;\xi,\eta,\zeta) &= \frac{1}{r} + \frac{1}{r_2} \\
&+ P.V. \int_0^\infty \frac{2(k+\upsilon)e^{-kh}\cosh(k(\zeta+h))\cosh(k(z+h))}{k\sinh kh - \upsilon\cosh kh} J_0(kR)dk \\
&- 2\pi i \frac{m_0^2-\upsilon^2}{(m_0^2-\upsilon^2)h+\upsilon}\cosh m_0(\zeta+h)\cosh m_0(z+h)J_0(m_0R). \qquad (7.188)
\end{aligned}$$

The second form of Green's function involves a series representation

$$\begin{aligned}
G(x,y,z;\xi,\eta,\zeta) &= 2\pi\frac{\upsilon^2-m_0^2}{(m_0^2-\upsilon^2)h+\upsilon}\cosh m_0(z+h) \\
&\times \cosh m_0(h+\zeta)[Y_0(m_0R)-iJ_0(m_0R)] + 4\sum_{k=1}^{\infty}\frac{(m_k^2+\upsilon^2)}{(m_k^2+\upsilon^2)h-\upsilon} \\
&\times \cos m_k(z+h)\cos m_k(h+\zeta)K_0(m_kR), \qquad (7.189)
\end{aligned}$$

for which $m_k, k > 0$ are the positive real roots of

$$m\tan mh + \upsilon = 0 \qquad (7.190)$$

We must find $Q(\xi,\eta,\zeta)$, the source density distribution function. This function must satisfy the third condition of equation (7.181), which applies at the body surface. The component of fluid velocity normal to the body surface must be equal to that of the structure itself, and may be expressed in terms of ϕ_S. Thus, following the treatment illustrated by Kellog [8], the normal derivative of the potential ϕ_S in equation (7.185) assumes the following form:

$$-2\pi Q(x,y,z) + \int\int_S Q(\xi,\eta,\zeta)\frac{\partial G}{\partial n}(x,y,z;\xi,\eta,\zeta)dS = -\frac{\partial\phi_I}{\partial n}(x,y,z)\times 4\pi \qquad (7.191)$$

This equation is known as Fredholm's integral equation of the second kind, which applies to the body surface S and must be solved for $Q(\xi,\eta,\zeta)$.

A suitable way of solving equation (7.191) for unknown source density Q is the matrix method. In this method, the surface of the body is discretized into a large number of panels, as shown in Figure 7.10. For each panel the source density is assumed to be constant. This replaces equation (7.191) by a set of linear algebraic equations with Q on each being an unknown.

These equations may be written as

$$\sum_{j=1}^{N} A_{ij}Q_j = a_i \quad \text{for } i = 1,2,\ldots,N \qquad (7.192)$$

where N is the number of panels.

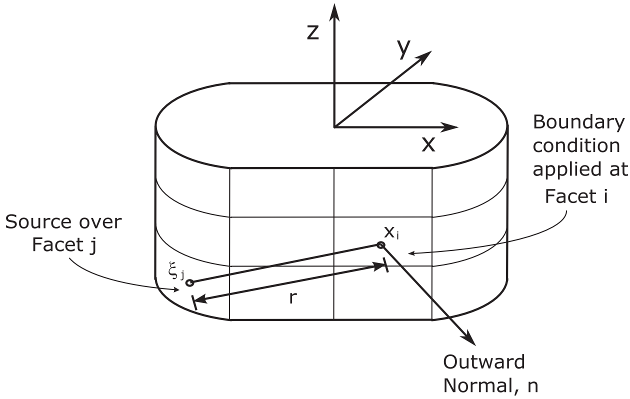

Figure 7.10: Boundary condition at panel i due to source density over panel j.

The coefficients a_i and A_{ij} are given, respectively, by

$$a_i = -2\frac{\partial \phi_I}{\partial n}(x_i, y_i, z_i) \tag{7.193}$$

$$A_{ij} = -\delta_{ij} + \frac{1}{2\pi}\int\int_{\Delta S_j} \frac{\partial G}{\partial n}(x_i, y_i, z_i; \xi_j, \eta_j, \zeta_j)dS \tag{7.194}$$

where δ_{ij} is the Kronecker delta, ($\delta_{ij} = 0$ for $i \neq j$, $\delta_{ii} = 1$), the point (x_i, y_i, z_i) is the centroid of the ith panel of area ΔS_i, and n is measured normal to the surface at that point. Assuming the value of $\frac{\partial G}{\partial n}$ is constant over the panel and equal to the value at the centroid, the expression for A_{ij} can be approximated as

$$\begin{aligned} A_{ij} &= -\delta_{ij} + \frac{\Delta S_j}{2\pi}\frac{\partial G}{\partial n}(x_i, y_i, z_i; \xi_j, \eta_j, \zeta_j) \\ &= -\delta_{ij} + \frac{\Delta S_j}{2\pi}\left[\frac{\partial G}{\partial x}n_x + \frac{\partial G}{\partial y}n_y + \frac{\partial G}{\partial z}n_z\right], \end{aligned} \tag{7.195}$$

in which n_x, n_y, and n_z are the unit normal vectors defining the panel orientation.

Note that when $i = j$, $\delta_{ii} = 1$ and the last term in equation (7.195) is equal to zero, and therefore may be omitted.

The column vector a_i in equation (7.193) may be evaluated as

$$a_i = -2\left[\frac{\partial \phi_I}{\partial x}n_x + \frac{\partial \phi_I}{\partial z}n_z\right], \tag{7.196}$$

where

$$\phi_I = \frac{gA}{\sigma}\cosh k(z+h)\cosh khe^{ikx}$$

Thus,

$$a_i = -2\left(\frac{gAk}{\sigma}\right)e^{ikx}\left[i\frac{\cosh k(z+h)}{\cosh kh}n_x + \frac{\sinh k(z+h)}{\cosh kh}n_z\right] \tag{7.197}$$

Once A_{ij} and a_i are known, the source distribution Q_j may be obtained by a complex matrix inversion procedure.

We then obtain the potential ϕ_S around the body surface by using a discrete version of equation (7.185) which is

$$\phi_S(x_i, y_i, z_i) = \sum_{j=1}^{\infty} B_{ij}Q_j, \tag{7.198}$$

where

$$\begin{aligned} B_{ij} &= \frac{1}{4\pi}\int\int_{\Delta S_j} G(x_i, y_i, z_i; \xi_j, \eta_j, \zeta_j)dS \\ &= \frac{\Delta S_j}{4\pi} G(x_i, y_i, z_i; \xi_j, \eta_j, \zeta_j). \end{aligned} \tag{7.199}$$

But when $i = j$, there exists a singularity in Green's function; however, we can still evaluate B_{ij} if we retain the dominant term in Green's function as follows:

$$B_{ij} = \frac{1}{4\pi}\int\int_{\Delta S_i} \frac{dS}{r}. \tag{7.200}$$

This may be integrated for a given panel.

Once the potentials, ϕ_S and ϕ_I, have been determined, we can obtain the pressure force P, surface elevation, η , forces, $\vec{F}$, and moments, $\vec{M}$, in a sequence.

The matrix equation (7.198) may be solved using a standard complex matrix inversion subroutine, such as that available in the IMSL library. The numerical prediction can then be compared with available experimental measurements. Hogben *et al* [6] have published a number of comparisons between computer predictions and experimental results. Figure 7.11 shows the comparisons of computed and measured forces and moments for both circular and square cylinders reported by Hogben and Standing [5]. More comprehensive published results dealing with offshore structures of arbitrary shapes can be found in the works of Van Oortmerssen [16], Faltinsen and Michelsen [1], Garrison and Rao [3], Garrison and Chow [4], and Mogridge and Jamieson [10].

Fenton [2] has applied Green's function to obtain the scattered wave field and forces on a body which is axisymmetric about a vertical axis. Starting with John's [7] equations, he expressed them as a Fourier series in terms of an azimuthal angle about the vertical axis of symmetry. He obtained a one-dimensional integral equation which was integrated analytically. Fenton [2] then applied his theory to a right

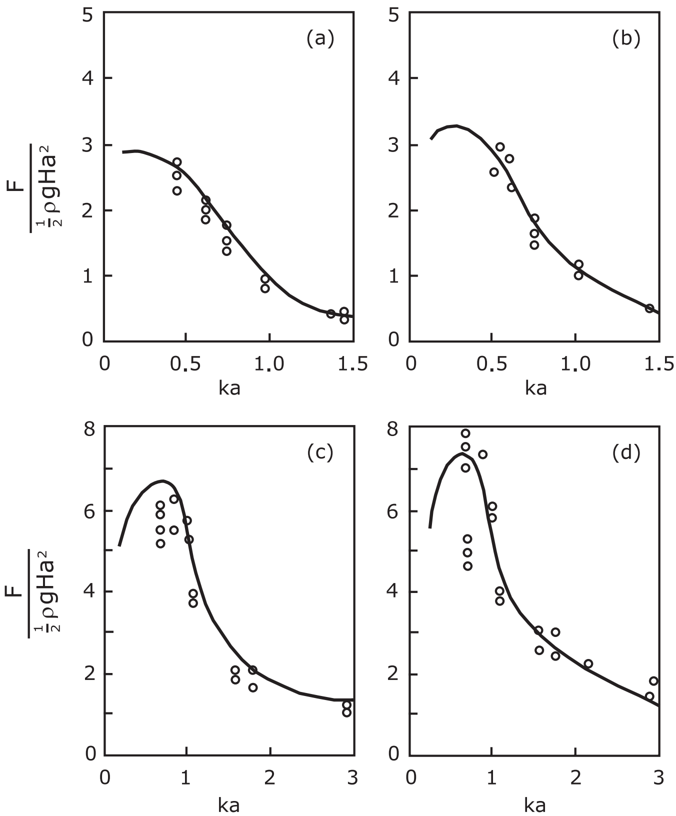

Figure 7.11: Comparison of computer prediction with experimental data of Hogben and Standing [5] for horizontal wave forces on circular and square columns. (a) circular, $h'/h = 0.7$, (b) square, $h'/h = 0.7$, (c) circular, surface-piercing, (d) square, surface-piercing. Here, h' is the height of the cylinder.

circular cylinder fixed to the ocean bed, in water of depth h; the cylinder was $0.7h$ and had a diameter of $0.4h$. He compared his theoretical computations with the experimental results of Hogben and Standing [5], and his results are reproduced in Figure 7.12. Fenton recorded that convergence of the Fourier series for Green's function was extremely rapid, and he was able to achieve an accuracy of 0.0001.

7.6 Remarks on symmetric kernel and a process of orthogonalization

For a symmetric kernel, that is, for a kernel for which

$$K(x,t) = K(t,x) \tag{7.201}$$

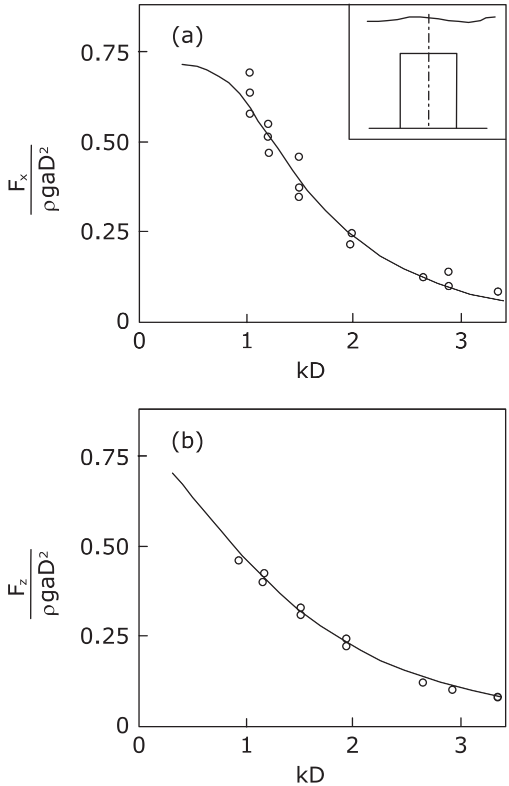

Figure 7.12: Variation of (a) dimensionless drag force and (b) dimensionless vertical force with wavenumber for a truncated circular cylinder of height $h' = 0.7h$ and diameter $D = 0.4h$, where h is the water depth. Experimental results from Hogben and Standing [5] are with dots, and Fenton's [2] computer predictions are with bold line.

the associated eigenfunctions ψ_h coincide with the proper eigenfunctions ϕ_h. It follows from the orthogonality property that any pair $\phi_h(x), \phi_k(x)$ of eigenfunctions of a symmetric kernel, corresponding to two different eigenvalues, λ_h, λ_k, satisfy a similar orthogonality condition

$$(\phi_h, \phi_k) \equiv \int_a^b \phi_h(x)\phi_k(x)dx = 0 \quad (h \neq k) \tag{7.202}$$

in the basic interval (a, b). Because of this connection between symmetric kernels and orthogonal systems of functions, namely

$$\{\phi_h\} \equiv \phi_1(x), \phi_2(x), \phi_3(x) \ldots \tag{7.203}$$

which satisfies equation (7.202), it is suitable to begin our study of the symmetric Fredholm integral equation with a brief survey of orthogonal functions.

We shall assume throughout that every function ϕ_h of our orthogonal systems is an L_2-function which does not vanish almost everywhere, i.e.

$$||\phi_h||^2 = \int_a^b \phi_h^2(x)dx > 0. \tag{7.204}$$

Thus, we can suppose that our functions are not only orthogonalized but also normalized, i.e.

$$(\phi_h, \phi_k) = \begin{Bmatrix} 0 & (h \neq k), \\ 1 & (h = k). \end{Bmatrix} \tag{7.205}$$

Such a system will be called orthonormal system. The functions of any orthonormal system are linearly independent; for, if there exist constants $c_1, c_2, c_3, \ldots c_n$ which are not all zero and are such that

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n \equiv 0, \tag{7.206}$$

almost everywhere in the basic interval (a,b), then multiplying by $\phi_h(x)(h = 1,2,3\ldots,n)$ and integrating over (a,b), we have

$$c_h \int_a^b \phi_h^2(x)dx = 0,$$

which, by equation (7.204), implies that $c_h = 0$, i.e. $c_1 = c_2 = \cdots = c - n = 0$. It is amazing that linear independence is not only necessary for orthogonality, but, in a certain sense, also sufficient. This is clear because we can always use the following procedure.

7.7 Process of orthogonalization

Given any finite or denumerable system of linearly independent L_2-functions $\psi_1(x), \psi_2(x), \psi_3(x), \ldots,$ it is always possible to find constants h_{rs} such that the functions

$$\begin{aligned}
\phi_1(x) &= \psi_1(x), \\
\phi_2(x) &= h_{21}\psi_1(x) + \psi_2(x), \\
\phi_3(x) &= h_{31}\psi_1(x) + h_{32}\psi_2(x) + \psi_3(x), \\
\cdots\cdots\cdots &= \cdots\cdots\cdots\cdots\cdots \\
\phi_n(x) &= h_{n1}\psi_1(x) + h_{n2}\psi_2(x) + \cdots + h_{nn-1}\psi_{n-1}(x) + \psi_n(x), \\
\cdots\cdots\cdots &= \cdots\cdots\cdots\cdots
\end{aligned} \tag{7.207}$$

are orthogonal in the basic interval (a, b). We prove this by mathematical induction. Observe first that the system (7.207) can be readily resolved with respect to the functions $\psi_1, \psi_2, \ldots$, i.e. it can be put to the equivalent form

$$
\begin{aligned}
\phi_1(x) &= \psi_1(x), \\
\phi_2(x) &= k_{21}\phi_1(x) + \psi_2(x), \\
\phi_3(x) &= k_{31}\phi_1(x) + k_{32}\phi_2(x) + \psi_3(x), \\
\cdots\cdots\cdots &= \cdots\cdots\cdots\cdots\cdots \\
\phi_n(x) &= k_{n1}\phi_1(x) + k_{n2}\phi_2(x) + \cdots + k_{nn-1}\phi_{n-1}(x) + \psi_n(x), \\
\cdots\cdots\cdots &= \cdots\cdots\cdots\cdots
\end{aligned}
\tag{7.208}
$$

We shall suppose that for $n-1$ functions the coefficients k_{rs} have already been determined i.e. we know k_{rs} for $1 \leq s < r \leq n-1$. We shall now show that the coefficients for the nth function k_{rs} $(s = 1, 2, 3, \ldots, n-1)$ can be readily calculated. In fact, from the $n-1$ conditions

$$
\begin{aligned}
0 &= (\phi_n, \phi_s) \\
&= k_{n1}(\phi_1, \phi_s) + k_{n2}(\phi_2, \phi_s) + \cdots + k_{nn-1}(\phi_{n-1}, \phi_s) + (\psi_n, \phi_s) \\
&= k_{ns}(\phi_s, \phi_s) + (\psi_n, \phi_s) \quad (s = 1, 2, \ldots, n-1)
\end{aligned}
$$

and we get

$$
k_{ns} = -\frac{(\psi_n, \phi_s)}{(\phi_s, \phi_s)}. \tag{7.209}
$$

These coefficients are well defined; $(\phi_s, \phi_s) \neq 0$ because ϕ_s is a linear combination of the linearly independent functions $\psi_1, \psi_2, \ldots, \psi_n$ and hence cannot be equal to zero almost everywhere. We illustrate the theory by an example below.

Example 7.10

Determine the orthogonal set of functions $\{\phi_n\}$ given that $\psi_1(x) = 1, \psi_2(x) = x$, $\psi_3(x) = x^2, \psi_4(x) = x^3 \ldots, \psi_n(x) = x^{n-1}$ defined on the interval $(0, 1)$.

Solution

The orthogonal set of functions is defined by equation (7.208) and its coefficients can be obtained by using equation (7.209). And hence we obtain using $\phi_1(x) = \psi_1(x) = 1$ the coefficient

$$
k_{21} = -\frac{(\psi_2, \phi_1)}{(\phi_1, \phi_1)} = -\frac{1}{2}.
$$

Thus, $\phi_2(x) = x - \frac{1}{2}$. With this value of $\phi_2(x)$ we are in a position to determine the coefficients k_{31} and k_{32}. They are given by

$$k_{31} = -\frac{(\psi_3, \phi_1)}{(\phi_1, \phi_1)} = -\frac{1}{3}$$

$$k_{32} = -\frac{(\psi_3, \phi_2)}{(\phi_2, \phi_2)} = -1$$

Thus, the expression for $\phi_3(x)$ can be obtained as $\phi_3(x) = x^2 - x + \frac{1}{6}$. The process of orthogonalization can be continued in this manner. It can be easily verified that the set of functions $\phi_1(x)$, $\phi_2(x)$, and $\phi_3(x)$ are orthogonal over the interval (0, 1).

Example 7.11

Show that the boundary value problem $y''(x) + \lambda y(x) = 0$ with the boundary conditions $y(0) = 0$, $y(1) = 0$ has the eigenfunction $\{y_n(x)\} = B_n \sin n^2\pi^2 x$ for $n = 1, 2, 3, \ldots$ and this set of functions are orthogonal over the interval (0, 1). Verify that the kernel associated with this boundary value problem are symmetric.

Solution

The general solution of this differential equation for $\lambda > 0$ is simply $y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. Using the boundary conditions its solution can be written as the eigenfunction $\{y_n(x)\} = B_n \sin(n\pi x)$ for $n = 1, 2, 3\ldots$, where the eigenvalues are given by $\lambda_n = n^2\pi^2$ for $n = 1, 2, 3, \ldots$. These solutions are orthogonal set of functions over the interval (0, 1).

The given differential equation with its boundary conditions can be transformed into an Fredholm integral equation. This can be accomplished by integrating two times with respect to x from 0 to x and using the condition at $x = 0$ yields

$$y(x) + \lambda \int_0^x \int_0^x y(t)dtdt = xy'(0).$$

This can be reduced to the following after using the condition at $x = 1$ and replacing the double integrals by a single integral

$$y(x) + \lambda \int_0^x (x - t)y(t)dt = \lambda x \int_0^1 (1 - t)y(t)dt$$

which can be very simply written in the integral form

$$y(x) = \lambda \int_0^1 K(x, t)y(t)dt,$$

where $K(x, t)$ is called the kernel of the equation and is given by

$$K(x, t) = \begin{cases} (1 - x)t & 0 \le t \le x \le 1 \\ (1 - t)x & 0 \le x \le t \le 1. \end{cases}$$

It can be easily verified that $K(x,t) = K(t,x)$ indicating that the kernel associated with this integral equation is symmetric.

7.8 The problem of vibrating string: wave equation

The partial differential equation that governs the vibration of an elastic string is given by

$$\eta_{tt} = C^2 \eta_{xx} \tag{7.210}$$

The boundary and initial conditions associated with this problem are

$$\begin{aligned} \eta(0,t) &= 0, \\ \eta(1,t) &= 0; \end{aligned} \tag{7.211}$$

$$\begin{aligned} \eta(x,0) &= f(x), \\ \eta_t(x,0) &= g(x) \end{aligned} \tag{7.212}$$

where

$$f(0) = 0, f(1) = 0$$
$$g(0) = 0, \ g(1) = 0.$$

C is the real positive constant and physically it is the speed of the wavefront. For general case, C can be a function of (x,t) also. In that case the string is not homogeneous.

Using the separation of variables method, we try for a solution of the form $\eta(x,t) = u(x)\phi(t)$. Substitution of this expression for η in equation (7.210) gives

$$u(x)\phi''(t) = C^2 u''(x)\phi(t)$$

which can be separated as

$$\frac{\phi''(t)}{C^2\phi(t)} = \frac{u''(x)}{u(x)}.$$

But the left-hand side is a function of t alone, the right-hand side member is a function of x alone; they are equal, hence equal to the same constant, say $-\lambda$. We are thus led to two ordinary differential equations:

$$u''(x) + \lambda u(x) = 0 \tag{7.213}$$

with the boundary conditions

$$u(0) = 0, u(1) = 0, \tag{7.214}$$

and

$$\phi''(t) + C^2\lambda\phi(t) = 0 \tag{7.215}$$

with the initial conditions

$$\eta(x,0) = u(x)\phi(0) = f(x), \eta_t(x,0) = u(x)\phi_t(0) = g(x) \tag{7.216}$$

For the boundary value problem, i.e. in equations (7.213) and (7.214), all of the characteristic constants are positive. Green's function $K(x,t)$ of this problem can be written at once

$$K(x,t) = \begin{cases} (1-x)t & 0 \leq t \leq x \leq 1 \\ (1-t)x & 0 \leq x \leq t \leq 1. \end{cases}$$

The boundary value problem, i.e. equations (7.213) and (7.214) is equivalent to

$$u(x) = \lambda \int_0^1 K(x,t)u(t)dt.$$

In this integral the kernel is symmetric. This boundary value problem has an infinitude of real positive characteristic constants, forming an increasing sequence:

$$0 < \lambda_1 < \lambda_2 < \cdots$$

with corresponding normalized fundamental functions

$$u_n(x) = \sqrt{2}\sin(n\pi x),$$

for $n = 1, 2, 3, \ldots$.

We return now to equation (7.216) with $\lambda = \lambda_n$:

$$\phi''(t) + C^2\lambda_n\phi(t) = 0. \tag{7.217}$$

Since $\lambda_n > 0$, the general solution of equation (7.217) is

$$\phi(t) = A_n \cos(C\sqrt{\lambda_n}t) + B_n \sin(C\sqrt{\lambda_n}t).$$

Therefore, a solution of equation (7.210) which satisfies equation (7.211) is

$$\begin{aligned}\eta(x,t) &= \sum_{n=1}^{\infty} (A_n \cos(C\sqrt{\lambda_n}t) + B_n \sin(C\sqrt{\lambda_n}t))\, u_n(x) \\ &= \sum_{n=1}^{\infty} (A_n \cos(n\pi Ct) + B_n \sin(n\pi Ct))\sqrt{2}\sin(n\pi x). \end{aligned} \tag{7.218}$$

Now using the initial conditions at $t=0$, we obtain

$$f(x) = \sum_{n=1}^{\infty} \sqrt{2}A_n \sin(n\pi x)$$

$$g(x) = \sum_{n=1}^{\infty} \sqrt{2}n\pi C B_n \sin(n\pi x)$$

These are sine series for $f(x)$ and $g(x)$. For the development of an arbitrary function in trigonometric series we need to know only that the function is continuous and has a finite number of maxima and minima. These conditions are not so strong as those obtained by means of the theory of integral equations which were demanded for the development in series of fundamental functions. The coefficients A_n and B_n can be obtained as

$$A_n = \int_0^1 f(x)(\sqrt{2}\sin(n\pi x))dx;$$

$$B_n = \frac{1}{n\pi C}\int_0^1 g(x)(\sqrt{2}\sin(n\pi x))dx.$$

The time period $T_1 = \frac{2}{C}$ is the fundamental period of vibration of the string. The general period $T_n = \frac{2}{nC} = \frac{T_1}{n}$, and the amplitude is $\sqrt{A_n^2 + B_n^2}$. Upon the intensity of the different harmonics depends the quality of the tone. The tone of period $\frac{T}{n}$ is called the nth harmonic overtone, or simply the nth harmonic. For the nonhomogeneous string the period is $\frac{2\pi}{\sqrt{\lambda}}$.

7.9 Vibrations of a heavy hanging cable

Let us consider a heavy rope of length, $AB = 1$ (see Figure 7.13) suspended at one end A. It is given a small initial displacement in a vertical plane through AB' and then each particle is given an initial velocity. The rope is suspended to vibrate in a given vertical plane and the displacement is so small that each particle is supposed to move horizontally; the cross-section is constant; the density is constant; the cross-section is infinitesimal compared to the length.

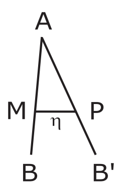

Figure 7.13: Vibrations of a heavy hanging cable.

Let AB' be the position for the rope at time t and P any point on AB'. Draw PM horizontal and put $MP = \eta$, $BM = x$.

Then the differential equation of the motion is given by

$$\frac{\partial^2 \eta}{\partial t^2} = C^2 \frac{\partial}{\partial x}\left(x \frac{\partial \eta}{\partial x}\right) \tag{7.219}$$

where C^2 is a constant, with the boundary conditions

$$\eta(1,t) = 0, \quad \eta(0,t) \text{ is finite} \tag{7.220}$$

$$\eta(x,0) = f(x), \quad \eta_t(x,0) = g(x). \tag{7.221}$$

By the separation of variables method, we try for a solution of the form

$$\eta(x,t) = u(x)\phi(t).$$

Substituting this expression for η in equation (7.219), we obtain

$$u(x)\phi''(t) = C^2\phi(t)\frac{d}{dx}\left(x\frac{du}{dx}\right).$$

This can be separated as

$$\frac{\phi''(t)}{C^2\phi(t)} = \frac{\frac{d}{dx}\left(x\frac{du}{dx}\right)}{u(x)} = -\lambda, \quad \text{constant.}$$

That is

$$\phi''(t) + C^2\lambda\phi(t) = 0,$$

and

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \lambda u = 0 \tag{7.222}$$

and the boundary conditions derived from equation (7.220);

$$u(0) \quad \text{finite} \quad u(1) = 0. \tag{7.223}$$

The differential equation (7.222) can be solved using the transformation $x = \frac{z^2}{4\lambda}$ and the equation reduces to the well-known Bessel equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + u = 0. \tag{7.224}$$

The general solution of this equation is

$$u(x) = C_1 J_0(2\sqrt{\lambda x}) + C_2 Y_0(2\sqrt{\lambda x}), \tag{7.225}$$

where C_1 and C_2 are two arbitrary constants. Here, J_0 and Y_0 are the zeroth-order Bessel functions of first and second kind, respectively. Since the solution must be finite at $x=0$, the second solution Y_0 can be discarded because it becomes infinite at $x=0$. Therefore, we have

$$u(x) = C_1 J_0(2\sqrt{\lambda x})$$

is the most general solution of equation (7.222), which satisfies the first initial condition. We have the further condition $u(1)=0$, hence

$$J_0(2\sqrt{\lambda}) = 0.$$

The solution of the equation gives us the eigenvalues. We know corresponding to each eigenvalue, there is an eigenfunction and the set of eigenfunctions are

$$u_n(x) = C_n J_0(2\sqrt{\lambda_n x})$$

for $n=1,2,3,\ldots$. There are infinitely many eigenfunctions as there are infinitely many eigenvalues. These eigenfunctions are defined over the interval (0, 1) and they are orthogonal.

We next construct the Green's function $K(x,t)$ for the boundary value problem, i.e. equations (7.222) and (7.223) satisfying the following conditions:

(a) K is continuous on (0,1).
(b) K satisfies the differential equation $\frac{d}{dx}\left(x\frac{dK}{dx}\right) = \delta(x-t)$ on (0,1).
(c) $K(0,t)$ finite, $K(1,t)=0$.
(d) $K'(t+0) - K'(t-0) = \frac{1}{t}$.

Integrating the differential equation in (b), we obtain

$$K(x,t) = \begin{cases} \alpha_0 \ln x + \beta_0, & (0,t) \\ \alpha_1 \ln x + \beta_1, & (t,1) \end{cases}$$

But $K(0,t)$ is finite, therefore $\alpha_0 = 0$, and $K(1,t)=0$, therefore $\beta_1 = 0$. Hence we have

$$K(x,t) = \begin{cases} K_0(x,t) = \beta_0, & (0,t) \\ K_1(x,t) = \alpha_1 \ln x, & (t,1) \end{cases}$$

From condition (a) $\beta_0 = \alpha_1 \ln t$. Also from the condition (d), since $K'(t-0)=0$ and $K'(t+0) = \frac{\alpha_1}{t}$, we obtain $\alpha_1 = 1$. Therefore,

$$K(x,t) = \begin{cases} K_0(x,t) = \ln t, & (0,t) \\ K_1(x,t) = \ln x, & (t,1) \end{cases} \tag{7.226}$$

We observe that $K(x,t)$ is symmetric.

Equivalence with a homogeneous integral equation

We now form the integral equation that will be equivalent to the differential equation (7.222)

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \lambda u = 0 \tag{7.227}$$

In differential operator symbol L we can write the above equation with the kernel equation as

$$L(u) = -\lambda u$$
$$L(K) = 0$$

Multiplying first of these by $-K$ and the second by u and adding we have

$$uL(K) - KL(u) = \lambda uK.$$

This equation can be explicitly written as

$$\frac{d}{dx}\{x(uK' - u'K)\} = \lambda Ku.$$

Integrate both members of this expression from $x = 0$ to $x = t - 0$ and from $x = t + 0$ to $x = 1$ with respect to x.

$$x(u(x)K'(x,t) - u'(x)K(x,t))|_{x=0}^{x=t-0} = \lambda \int_{x=0}^{x=t-0} K(x,t)u(x)dx$$

$$x(u(x)K'(x,t) - u'(x)K(x,t))|_{x=t+0}^{x=1} = \lambda \int_{x=t+0}^{x=1} K(x,t)u(x)dx$$

Adding these two equations yields

$$\left[x(uK' - u'K)\right]_{x=t+0}^{x=t-0} + \left[x(uK' - u'K)\right]_{x=0}^{x=1} = \lambda \int_0^1 K(x,t)u(x)dx \tag{7.228}$$

The second term of the first expression of equation (7.228) on the left-hand side is zero because of the continuity of K. That means $K(t-0) = K(t+0) = K(t)$. But the first term yields

$$\begin{aligned}
\left[x(uK')\right]_{x=t+0}^{x=t-0} &= (t-0)u(t-0)K'(t-0) - (t+0)u(t+0)K'(t+0) \\
&= tu(t)[K'(t-0) - K'(t+0)] \\
&= tu(t)\frac{-1}{t} \\
&= -u(t)
\end{aligned}$$

The second term on the left-hand side of equation (7.228) is zero because $u(0)=0, u(1)=0$; and $K(0)=0$, $K(1)=0$.

Therefore the integral equation can be written as

$$u(t) = -\lambda \int_0^1 K(x,t)u(x)dx \tag{7.229}$$

Hence, changing the role of x and t, and that the kernel is symmetric, we can write the above equation as

$$u(x) = -\lambda \int_0^1 K(x,t)u(t)dt \tag{7.230}$$

This is the Fredholm integral equation which is equivalent to the ordinary differential equation (7.227).

Now coming back to the original problem, the eigenvalues will be given by the roots of $J_0(2\sqrt{\lambda})=0$, and these roots can be defined from $k_n = 2\sqrt{\lambda_n}$. The first four values of k_n are

$$k_1 = 2.405, k_2 = 5.520, k_3 = 8.654, k_4 = 11.792,$$

and generally $\left(n-\frac{1}{2}\right)\pi < k_n < n\pi$. Therefore,

$$u_n(x) = C_1 J_0(2\sqrt{\lambda_n x}) = C_1 J_0(k_n\sqrt{x}).$$

These fundamental functions $u_n(x)$ will become orthogonalized if we choose

$$C_1 = \frac{1}{\sqrt{\int_0^1 J_0^2(k_n\sqrt{x})dx}}.$$

But $\int_0^1 J_0^2(k_n\sqrt{x})dx = \frac{1}{[J_0'(k_n)]^2}$. Therefore,

$$u_n(x) = \frac{J_0(k_n\sqrt{x})}{J_0'(k_n)}$$

$$\eta(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{Ck_n t}{2} + B_n \sin\frac{Ck_n t}{2}\right)u_n(x).$$

This expression for $\eta(x,t)$ satisfies equations (7.219) and (7.220). We now determine A_n and B_n, if possible, in order that equation (7.221) may be satisfied. This gives us the two equations

$$\sum_{n=1}^{\infty} A_n u_n(x) = f(x)$$

$$\sum_{n=1}^{\infty} \frac{Ck_n}{2} B_n u_n(x) = g(x)$$

Because the series on the left-hand side is composed of the set of orthogonal functions, we can evaluate the coefficients A_n and B_n as follows:

$$A_n = \int_0^1 f(x)u_n(x)dx = \int_0^1 f(x)\frac{J_0(k_n\sqrt{x})}{J_0'(k_n)}dx,$$

$$B_n = \frac{2}{Ck_n}\int_0^1 g(x)u_n(x)dx = \frac{2}{Ck_n}\int_0^1 g(x)\frac{J_0(k_n\sqrt{x})}{J_0'(k_n)}dx.$$

Once these coefficients are determined the problem is completely solved.

7.10 The motion of a rotating cable

With reference to Figure 7.14, let FG be an axis around which a plane is rotating with constant velocity; a cable AB is attached at a point A of the axis and constrained to remain in the rotating plane. The velocity of the rotation is so large that the weight of the cable can be neglected. Then the straight line AB perpendicular to FG is a relative position of equilibrium for the cable. Displace the cable slightly from this position AB, then let it go after imparting to its particles initial velocities perpendicular to AB. The cable will then describe small vibrations around the position of equilibrium.

Let APB' be the position of the cable at the time t, P one of its points, PM is perpendicular to AB. Put $AM = x$, $MP = \eta$, and suppose $AB = 1$. Then the function $\eta(x,t)$ must satisfy the partial differential equation

$$\frac{\partial^2\eta}{\partial t^2} = C^2\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial\eta}{\partial x}\right], \tag{7.231}$$

where C is a constant. The boundary conditions are

$$\begin{aligned}\eta(0,t) &= 0\\ \eta(1,t) &= \text{finite}.\end{aligned} \tag{7.232}$$

The initial conditions are

$$\begin{aligned}\eta(x,0) &= f(x)\\ \eta_t(x,0) &= g(x).\end{aligned} \tag{7.233}$$

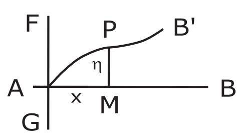

Figure 7.14: The motion of a rotating cable.

Putting $\eta(x,t) = u(x)\phi(t)$, we obtain two ordinary differential equations

$$\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right] + \lambda u = 0 \tag{7.234}$$

with boundary conditions $u(0) = 0$, $u(1) = \text{finite}$. The $\phi(t)$ satisfies

$$\frac{d^2\phi}{dt^2} + \lambda C^2\phi = 0. \tag{7.235}$$

The general solution of the equation (7.234) can be written at once if the eigenvalues $\lambda_n = n(n+1)$ for $n = 0, 1, 2, 3, \ldots$ as the Legendre's polynomials $P_n(x)$ and $Q_n(x)$

$$u(x) = AP_n(x) + BQ_n(x) \tag{7.236}$$

where A and B are two arbitrary constants. Here, the second solution is obtained from the Abel's identity

$$Q_n(x) = P_n(x)\int \frac{dx}{(1-x^2)P_n^2(x)}. \tag{7.237}$$

We can at once indicate some of the eigenvalues and the corresponding fundamental eigenfunctions from the theory of Legendre's polynomials:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\{(x^2-1)^n\}.$$

Using this formula, we can evaluate the different components of the Legendre's polynomials

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\
P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \\
\cdots\cdots &= \cdots\cdots
\end{aligned}$$

We see that $P_n(1) = 1$, and $P_n(-1) = (-1)^n$ for all values of n. The Legendre polynomial $P_n(x)$ is finite in the closed interval $-1 \le x \le 1$.

Hence $Q_0(x)$ and $Q_1(x)$ can be very easily calculated as

$$Q_0(x) = \int \frac{dx}{1-x^2} = \frac{1}{2}\ln\frac{1+x}{1-x}$$

$$Q_1(x) = x\int \frac{dx}{(1-x^2)x^2} = -1 + x\ln\frac{1+x}{1-x}.$$

Thus, it is easy to see that $Q_n(x)$ is a singular solution at $x = \pm 1$, and hence $B = 0$. The solution, i.e. equation (7.236) takes the form

$$u(x) = AP_n(x). \tag{7.238}$$

$P_n(x)$ is a rational integral function of degree n, satisfying the differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right] + n(n+1)u = 0.$$

Furthermore, P_{2n} is an even function and $P_{2n}(0) \neq 0$; however, $P_{2n-1}(x)$ is an odd function and $P_{2n-1}(0) = 0$. Therefore, $\lambda_n = 2n(2n-1)$ is an eigenvalue and $P_{2n-1}(x)$ a corresponding fundamental eigenfunction. Therefore, only the odd Legendre's polynomials will satisfy the given boundary condition of the present problem. It is worth noting that the Legendre polynomials form an orthogonal set of functions defined in the closed interval $-1 \leq x \leq 1$. Therefore, the orthogonality property is satisfied, that means

$$\int_{-1}^{1} P_{2n-1}(x)P_{2m-1}(x)dx = 0,$$

for $m \neq n$. Or more explicitly, this orthogonality property is valid even for the interval $0 \leq x \leq 1$ such that

$$\int_{0}^{1} P_{2n-1}(x)P_{2m-1}(x)dx = 0,$$

for $m \neq n$. Thus, the orthonormal set of functions can be constructed as

$$\varphi_n(x) = \frac{P_{2n-1}(x)}{\sqrt{\int_0^1 P_{2n-1}^2(x)dx}},$$

for $n = 1, 2, 3 \ldots$. But

$$\int_0^1 P_{2n-1}^2(x)dx = \frac{1}{4n-1},$$

as shown in the theory of Legendre's polynomials. Therefore,

$$\varphi_n(x) = \sqrt{4n-1}P_{2n-1}(x),$$

is the orthonormal set of Legendre's polynomials.

Equivalence with integral equation

We construct the Green's function as before and obtain

$$K(x,t) = \begin{cases} \frac{1}{2}\ln\frac{1+x}{1-x}, & 0 \le x \le t \\ \frac{1}{2}\ln\frac{1+x}{1-x}, & t \le x \le 1. \end{cases}$$

$K(x, T)$ is symmetric, but has one point of discontinuity and that is for $x = t = 1$. Proceeding as in the previous problem, we find that the boundary value problem is equivalent to the integral equation:

$$u(x) = \lambda \int_0^1 K(x,t)u(t)dt.$$

Now, the final solution of this rotating cable problem can be obtained using the initial conditions and then evaluating the unknown constants. The final general solution is

$$\eta(x,t) = \sum_{n=1}^{\infty} [A_n \cos(C\sqrt{\lambda_n}t) + B_n \sin(C\sqrt{\lambda_n}t)]P_{2n-1}(x). \tag{7.239}$$

Using the initial conditions, we obtain

$$f(x) = \sum_{n=1}^{\infty} A_n P_{2n-1}(x)$$

$$g(x) = \sum_{n=1}^{\infty} (C\sqrt{\lambda_n})B_n P_{2n-1}(x)$$

where the coefficients A_n and B_n are obtained

$$A_n = (4n-1)\int_0^1 f(x)P_{2n-1}(x)dx$$

$$B_n = \frac{(4n-1)}{C\sqrt{\lambda_n}}\int_0^1 g(x)P_{2n-1}(x)dx.$$

Thus, the problem is completely solved.

7.11 Exercises

1. Show that the solution of

$$y'' = f(x), y(0) = y(1) = 0 \quad \text{is}$$

$$y = \int_0^1 G(x,\eta)f(\eta)d\eta$$

where

$$G(x,\eta) = \begin{cases} \eta(x-1) & 0 \le \eta \le x \\ x(\eta-1) & x \le \eta \le 1. \end{cases}$$

2. Discuss how you might obtain $G(x,\eta)$ if it were not given.
[Hint: One possibility is to write

$$y = \int_0^x G(x,\eta)f(\eta)d\eta + \int_x^1 G(x,\eta)f(\eta)d\eta$$

and substitute into the given equation and boundary conditions to find suitable conditions on G in the two regions $0 \le \eta \le x, x \le \eta \le 1$.]

3. Apply your method in Exercise 2 to solve

$$y'' + y = f(x), \quad y(0) = y(1) = 0.$$

4. Find the deflection curve of a string of length ℓ bearing a load per unit length $W(x) = -x$, first by solving the differential equation $Ty'' = -x$ with the boundary conditions $y(0) = y(\ell) = 0$ and then by using the Green's function for the string.

5. Construct the Green's function for the equation $y'' + 2y' + 2y = 0$ with the boundary conditions $y(0) = 0$, $y(\frac{\pi}{2}) = 0$. Is this Green's function symmetric? What is the Green's function if the differential equation is

$$e^{2x}y'' + 2e^{2x}y' + 2e^{2x}y = 0?$$

Is this Green's function symmetric?

Find the Green's function for each of the following boundary value problems:

6. $y'' + y = 0; \quad y(0) = 0,\ y'(\pi) = 0.$
7. $y'' + y' = 0; \quad y(0) = 0,\ y'(\pi) = 0.$
8. $y'' = 0; \quad y(0) = 0,\ y(1) = 0.$
9. $y'' = 0; \quad y(0) = 0,\ y'(1) = 0.$
10. $y'' + \lambda^2 y = 0; \quad \lambda \ne 0, y(0) = y(1),\ y'(0) = y'(1).$
11. $y'' + \lambda^2 y = 0; \quad \lambda \ne 0$, if the boundary conditions are

    (a) $y(0) = y'(b) = 0,$
    (b) $y'(0) = y(b) = 0,$
    (c) $y'(a) = y'(b) = 0,$
    (d) $y(a) = y'(a), y(b) = 0$

12. Find all Green's functions corresponding to the equation $x^2y'' - 2xy' + 2y = 0$ with the boundary conditions $y(0) = y(1)$. Why does this differential system have infinitely many Green's functions? [Hint: Where is the differential equation normal?]

13. Find the Green's functions for the following problems:

$$(a) \quad (1 - x^2)y'' - 2xy' = 0; \quad y(0) = 0, y'(1) = 0.$$
$$(b) \quad y'' + \lambda^2 y = 0; \quad y(0) = 0, y(1) = 0$$

14. Show that the Green's function $G(t, \eta)$ for the forced harmonic oscillation described by the initial value problem

$$\ddot{y} + \lambda^2 y = A \sin \omega t; \quad y(0) = 0, \dot{y}(0) = 0$$

is $G(t, \eta) = \frac{1}{\lambda} \sin \lambda(t - \eta)$.
Here, A is a given constant. Hence the particular solution can be determined as

$$y = \frac{A}{\lambda} \int_0^t \sin \lambda(t - \eta) \sin \omega \eta d\eta$$

15. Determine the Green's function for the boundary value problem

$$y'' = -f(x); \quad y(-1) = y(1), y'(-1) = y'(1)$$

16. Determine the Green's function for the boundary value problem

$$xy'' + y' = -f(x)$$
$$y(1) = 0, \quad \lim_{x \to 0} |y(x)| < \infty.$$

17. By using the Green's function method solve the boundary value problem $y'' + y = -1; y(0) = y\left(\frac{\pi}{2}\right) = 0$. Verify your result by the elementary technique.

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8 Applications

8.1 Introduction

The development of science has led to the formation of many physical laws, which, when restated in mathematical form, often appear as differential equations. Engineering problems can be mathematically described by differential equations, and thus differential equations play very important roles in the solution of practical problems. For example, Newton's law, stating that the rate of change of the momentum of a particle is equal to the force acting on it, can be translated into mathematical language as a differential equation. Similarly, problems arising in electric circuits, chemical kinetics, and transfer of heat in a medium can all be represented mathematically as differential equations. These differential equations can be transformed to the equivalent integral equations of Volterra and Fredholm types. There are many physical problems that are governed by the integral equations and these equations can be easily transformed to the differential equations. This chapter will examine a few physical problems that lead to the integral equations. Analytical and numerical methods will be illustrated in this chapter.

8.2 Ocean waves

A remarkable property of wave trains is the weakness of their mutual interactions. From the dynamic point of view, the interaction of waves causes the energy transfer among different wave components. To determine the approximate solution of a nonlinear transfer action function we consider a set of four progressive waves travelling with different wave numbers and frequencies and this set, which is called a quadruple, could exchange energy if they interact nonlinearly. Three analytical methods due to Picard, and Adomian, and one numerical integration method using a fourth-order Runge−Kutta scheme are used to compute the nonlinear transfer action function. Initial conditions are used from the JONSWAP (Joint North Sea Wave Project) spectrum. The results obtained from these four methods are compared in graphical forms and we have found excellent agreement among them.

8.2.1 Introduction

The wave–wave interactions of four progressive waves travelling with four different wave numbers and frequencies has been studied extensively by many previous researchers including Webb [22], Phillips [14], and Hasselmann [5]. This set of four waves, called a quadruple, could exchange energy if they interact nonlinearly such that resonant conditions are satisfied.

In the present work, we shall proceed with the approximate solution of nonlinear transfer action functions that satisfy a set of first-order ordinary differential equations using the JONSWAP spectrum as initial conditions. For this situation, we consider the energy balance equation by Phillips [2] to show the various physical processes that cause the change of energy of a propagating wave group. For the input energy due to wind, we take the parameterizations proposed by Hasselmann *et al* in the WAM model (see Refs. [6], [9], and [10]). The empirical formulation of the change of energy due to wave interaction with ice floes has been described by Isaacson [8], Masson and LeBlond [12] within MIZ (marginal ice zone). For nonlinear transfer between the spectral components, we take the parameterizations proposed by Hasselmann *et al* [6] where the energy spectrum is actually proportional to the action spectrum and the proportionality constant is given by the radian frequency ω. Four simple methods are demonstrated in this section to compare the nonlinear transfer action function. Three of them are analytic due to Picard, Bernoulli, and Adomian, one is numerical integration using a fourth-order Runge–Kutta scheme and the results are compared in graphical form.

8.2.2 Mathematical formulation

As suggested by Masson and LeBlond [8], the two-dimensional ocean wave spectrum $E(f,\theta;\mathbf{x},t)$ which is a function of frequency f (cycles/s, Hz) and θ, the direction of propagation of wave, time t, and position $\mathbf{x}$ satisfies the energy balance equation within MIZ, i.e.

$$\left(\frac{\partial}{\partial t} + \mathbf{C_g}\cdot\nabla\right)E(f,\theta) = (S_{\text{in}} + S_{\text{ds}})(1-f_{\text{i}}) + S_{\text{nl}} + S_{\text{ice}}, \tag{8.1}$$

where $\mathbf{C_g}$ is the group velocity and ∇ the gradient operator. The right-hand side of the above equation is a net source function describing the change of energy of a propagating wave group. Here, S_{in} is the input energy due to wind, S_{ds} is the energy dissipation due to white capping, S_{nl} is the nonlinear transfer between spectral components due to wave–wave interactions, S_{ice} is the change in energy due to wave interactions with ice floes and f_{i} is the fraction of area of the ocean covered by ice. Equation (8.1) can be written as

$$\frac{dE(f,\theta)}{dt} = (S_{\text{in}} + S_{\text{ds}})(1-f_{\text{i}}) + S_{\text{nl}} + S_{\text{ice}} \tag{8.2}$$

where $\frac{d}{dt}$ denotes the total differentiation when travelling with group velocity. The wind input source function S_{in} should be represented by

$$S_{\text{in}} = \beta E(f, \theta), \tag{8.3}$$

as parameterized in the WAM model of Hasselmann *et al* [6] and Komen *et al* [9] where

$$\beta = max\left\{0, 0.25\frac{\rho_a}{\rho_w}\left(28\frac{U_*}{C}\cos(\theta) - 1\right)\right\}\omega, \tag{8.4}$$

and $\omega = 2\pi f$ angular frequency. $\frac{\rho_a}{\rho_w}$ is the ratio of densities of air and water. θ is the angle between the wind vector and wave propagation direction. Many authors, including Komen *et al.* [10], have modified the formulation of β to simulate the coupling feedback between waves and wind. The dissipation source function used in WAM model is of the form (see Komen *et al* [9])

$$S_{\text{ds}} = -C_{\text{ds}}\left(\frac{\hat{\alpha}}{\hat{\alpha}_{PM}}\right)^2\left(\frac{\omega}{\overline{\omega}}\right)^2\overline{\omega}E(f, \theta) \tag{8.5}$$

wheret $\hat{\alpha} = m_0\overline{\omega}^4/g^2$, and m_0 is the zeroth moment of the variance spectrum and $\overline{\omega}$ is the mean radian frequency,

$$\overline{\omega} = \frac{\int\int E(\omega, \theta)d\omega d\theta}{E_{total}}, \tag{8.6}$$

in which

$$E_{total} = E(f, \theta)df\, d\theta \tag{8.7}$$

is the total spectral energy. Tuning is achieved by a filtering parameter, C_{ds} and $\hat{\alpha}/\hat{\alpha}_{pm}$ is the overall measure of steepness in the wave field. The empirical formulation for the change in energy due to wave interaction with ice floes, S_{ice}, has been described by Isaacson [8], Masson and LeBlond [12]. With the MIZ, the ice term S_{ice} is expressed in terms of a transformation tensor T_{fl}^{ij}

$$S(fl, \theta_i)_{\text{ice}} = E(f_e, \theta_j)T_{fl}^{ij} \tag{8.8}$$

where the space **x** and time t coordinates are important, and summation is over all j angle bands of discretization. The transformation tensor T_{fl}^{ij} is expressed as

$$T_{fl}^{ij} = A^2[\beta|D(\theta_{ij})|^2\Delta\theta + \delta(\theta_{ij})(1 + |\alpha_c D(0)|^2) + \delta(\pi - \theta_{ij})|\alpha_c D(\pi)|^2], \tag{8.9}$$

where δ is the Dirac delta function and $\Delta\theta$ is the angular increment in θ and $\theta_{ij} = |\theta_i - \theta_j|$. Other parameters arising in this formulation have been described by Masson and LeBlond [12], and will not be repeated here. The formulation of the nonlinear transfer between spectral components due to wave–wave interactions, S_{nl}, is rather complicated. It was demonstrated by Hasselmann *et al* [6] that the energy spectrum $E(f,\theta)$ is actually proportional to the action spectrum $N(f,\theta)$ such that $E(f,\theta) = \omega N(f,\theta)$, where the proportionality constant is the radian frequency ω. Hence, equation (8.2) can be written in two ways:

For energy:

$$\frac{dE}{dt} = (S_{\mathrm{nl}} + S_{\mathrm{ds}})_e(1 - f_{\mathrm{i}}) + (S_{\mathrm{nl}})_e + (S_{\mathrm{ice}})_e. \tag{8.10}$$

For action:

$$\frac{dN}{dt} = (S_{\mathrm{nl}} + S_{\mathrm{ds}})(1 - f_{\mathrm{i}}) + S_{\mathrm{nl}} + S_{\mathrm{ice}}. \tag{8.11}$$

Equation (8.11) is most basic because S_{nl} is expressed in terms of action. We shall proceed with the evaluation of the nonlinear wave–wave interaction S_{nl} with the use of the following approximate nonlinear simultaneous differential equations. Hasselmann *et al* [6] constructed a nonlinear interaction operator by considering only a small number of neighbouring and finite distance interactions. It was found that, in fact, the exact nonlinear transfer could be well simulated by just one mirror-image pair of intermediate-range interaction configurations. In each configuration, two wave numbers were taken as identical $\mathbf{k_1} = \mathbf{k_2} = \mathbf{k}$. The wave numbers $\mathbf{k_3}$ and $\mathbf{k_4}$ are of different magnitude and lie at an angle to the wave number $\mathbf{k}$, as required by resonance conditions. The second configuration is obtained from the first by reflecting the wave numbers $\mathbf{k_3}$ and $\mathbf{k_4}$ with respect to the $\mathbf{k}$ axis (see Ref. [9], p. 226). The scale and direction of the reference wave number are allowed to vary continuously in wave number space. For configurations

$$\begin{aligned} \omega_1 &= \omega_2 = \omega \\ \omega_3 &= \omega(1 + \lambda) = \omega_+ \\ \omega_4 &= \omega(1 - \lambda) = \omega_-, \end{aligned} \tag{8.12}$$

where $\lambda = 0.25$, a constant parameter, satisfactory agreement with exact computation was found. From the resonance conditions, the angles θ_3 and θ_4 of the wave numbers $\mathbf{k_3}(\mathbf{k_+})$ and $\mathbf{k_4}(\mathbf{k_-})$ relative to $\mathbf{k}$ are found to be $\theta_3 = 11.5°$, $\theta_4 = -33.6°$.

The discrete interaction approximation has its simplest form for deep ocean for the rate of change in time of action density in wave number space. The balance equation can be written as

$$\frac{d}{dt}\begin{Bmatrix} N \\ N_+ \\ N_- \end{Bmatrix} = \begin{Bmatrix} -2 \\ 1 \\ 1 \end{Bmatrix} Cg^{-8} f^{19}[N^2(N_+ + N_-) - 2NN_+N_-]\Delta\mathbf{k}, \tag{8.13}$$

where $\frac{dN}{dt}$, $\frac{dN_+}{dt}$, and $\frac{dN_-}{dt}$, are the rate of change in action at wave numbers $\mathbf{k}$, $\mathbf{k}_+$, and $\mathbf{k}_-$, respectively, due to the discrete interactions within the infinitesimal interaction phase-space element $\Delta\mathbf{k}$ and C is the numerical constant. The net source function S_{nl} can be obtained by summing equation (8.12) over all wave numbers, directions, and interaction configurations. Equation (8.12) is only valid for deep-water ocean. Numerical computations by Hasselmann and Hasselmann of the full Boltzmann integral for water of an arbitrary depth have shown that there is an approximate relationship between the transfer rate for deep-water and water of finite depth. For a frequency direction-spectrum, the transfer for a finite depth ocean is identical to the transfer of infinite depth, except for the scaling factor R:

$$S_{\mathrm{nl}}(\text{finite depth}) = R(\overline{k}h)S_{\mathrm{nl}}(\text{infinite depth}), \tag{8.14}$$

where $\overline{k}$ is the mean wave number. This scaling relation holds in the range $\overline{k}h > 1$, where the scaling factor can be expressed as

$$R(x) = 1 + \frac{5.5}{x}\left(1 - \frac{5x}{6}\right) exp\left(\frac{-5x}{4}\right), \tag{8.15}$$

with $x = \frac{3}{4}\overline{k}h$. The WAM model uses this approximation.

8.3 Nonlinear wave–wave interactions

This section will be devoted to the solution technique of the nonlinear wave–wave interactions. To determine the nonlinear wave–wave interaction S_{nl}, we can rewrite the equation (8.12) explicitly with their initial conditions in the following form

$$\frac{dN}{dt} = \alpha_1[N^2(N_+ + N_-) - 2NN_+N_-] \tag{8.16}$$

$$\frac{dN_+}{dt} = \alpha_2[N^2(N_+ + N_-) - 2NN_+N_-] \tag{8.17}$$

$$\frac{dN_-}{dt} = \alpha_3[N^2(N_+ + N_-) - 2NN_+N_-], \tag{8.18}$$

where

$$\begin{aligned}
\alpha_1 &= -2Cg^{-8}f^{19}\Delta k \\
\alpha_2 &= Cg^{-8}f^{19}\Delta k_+ \\
\alpha_3 &= Cg^{-8}f^{19}\Delta k_-.
\end{aligned} \tag{8.19}$$

The initial conditions are

$$\text{at} \qquad t = 0: \qquad \begin{aligned} N(0,f) &= N_0(f) \\ N_+(0,f) &= N_{+0}(f_+) \\ N_-(0,f) &= N_{-0}(f_-). \end{aligned} \tag{8.20}$$

The specific algebraic forms of these initial values will be stated later.

8.4 Picard's method of successive approximations

It can be easily observed that equations (8.16), (8.17), and (8.18) are related to each other as follows

$$\frac{dN_+}{dt} = \left(\frac{\alpha_2}{\alpha_1}\right)\frac{dN}{dt} \tag{8.21}$$

$$\frac{dN_-}{dt} = \left(\frac{\alpha_3}{\alpha_1}\right)\frac{dN}{dt}. \tag{8.22}$$

Thus, if we can determine the solution for $N(t,f)$, then solution for $N_+(t,f)$ and $N_-(t,f)$ can be easily determined by interaction from the equations (8.21) and (8.22). However, we shall integrate equations (8.16), (8.17), and (8.18) using Picard's successive approximation method.

8.4.1 First approximation

Since the equation is highly nonlinear, it is not an easy matter to integrate it at one time. In the first approximation, we shall replace the source terms on the right-hand side of the equation by their initial values, which will be integrated at once. For instance equation (8.16) can be written as

$$\frac{dN}{dt} = \alpha_1[N_0^2(N_{+0} + N_{-0}) - 2N_0N_{+0}N_{-0}]. \tag{8.23}$$

The right-hand side of equation (8.23) is a constant and can be integrated immediately, with respect to time from $t = 0$ to $t = t$:

$$N(t,f) = N_0(f) + \alpha_{10}t, \tag{8.24}$$

where

$$\alpha_{10} = \alpha_1[N_0^2(N_{+0} + N_{-0}) - 2N_0N_{+0}N_{-0}]. \tag{8.25}$$

Similarly, the solutions for equations (8.17) and (8.18) give

$$N_+(t,f) = N_{+0}(f_+) + \alpha_{20}t \tag{8.26}$$

$$N_-(t,f) = N_{-0}(f_-) + \alpha_{30}t, \tag{8.27}$$

where

$$\alpha_{20} = \alpha_2[N_0^2(N_{+0} + N_{-0}) - 2N_0N_{+0}N_{-0}] \tag{8.28}$$

$$\alpha_{30} = \alpha_3[N_0^2(N_{+0} + N_{-0}) - 2N_0N_{+0}N_{-0}]. \tag{8.29}$$

Equations (8.24), (8.26), and (8.27) are the first approximate solution.

8.4.2 Second approximation

To determine the second approximate solutions we have to update the source functions by the first approximate solutions and then integrate. For this we need to calculate the expression $\{N^2(N_+ + N_-) - 2N_+N_-\}$, and this gives

$$\begin{aligned} N^2(N_+ + N_-) - 2NN_+N_- &= [N_0^2(N_{+0} + N_{-0}) - 2N_0N_{+0}N_{-0}] \\ &\quad + [N_0^2(\alpha_{20} + \alpha_{30}) + 2\alpha_{10}N_0(N_{+0} + N_{-0}) \\ &\quad - 2\{N_0(\alpha_{30}N_{+0} + \alpha_{20}N_{-0}) + \alpha_{10}N_{+0}N_{-0}\}]t \\ &\quad + [2\alpha_{10}(\alpha_{20} + \alpha_{30})N_0 + \alpha_{10}^2(N_{+0} + N_{-0}) \\ &\quad -2\{\alpha_{20}\alpha_{30}N_0 + \alpha_{10}(\alpha_{30}N_{+0} + \alpha_{20}N_{-0})\}]t^2 \\ &\quad + [\alpha_{10}^2(\alpha_{20} + \alpha_{30}) - 2\alpha_{10}\alpha_{20}\alpha_{30}]t^3. \end{aligned} \tag{8.30}$$

Hence, the differential equation (8.16) can be written as

$$\begin{aligned} \frac{dN}{dt} &= \alpha_1[N^2(N_+ + N_-) - 2NN_+N_-] \\ &= a_0 + a_1t + a_2t^2 + a_3t^3, \end{aligned} \tag{8.31}$$

where

$$\begin{aligned} a_0 &= \alpha_1[N_0^2(N_{+0} + N_{-0}) - 2N_0N_{+0}N_{-0}] = \alpha_{10} \\ a_1 &= \alpha_1[N_0^2(\alpha_{20} + \alpha_{30}) + 2\alpha_{10}N_0(N_{+0} + N_{-0}) \\ &\quad - 2\{N_0(\alpha_{30}N_{+0} + \alpha_{20}N_{-0}) + \alpha_{10}N_{+0}N_{-0}\}] \\ a_2 &= \alpha_1[2\alpha_{10}(\alpha_{20} + \alpha_{30})N_0 + \alpha_{10}^2(N_{+0} + N_{-0}) \\ &\quad - 2\{\alpha_{20}\alpha_{30}N_0 + \alpha_{10}(\alpha_{30}N_{+0} + \alpha_{20}N_{-0})\}] \\ a_3 &= \alpha_1[\alpha_{10}^2(\alpha_{20} + \alpha_{30}) - 2\alpha_{10}\alpha_{20}\alpha_{30}]. \end{aligned} \tag{8.32}$$

Integrating equation (8.31) with respect to t from $t = 0$ to $t = t$, we obtain

$$N(t,f) = N_0(f) + a_0 t + a_1 \tfrac{t^2}{2} + a_2 \tfrac{t^3}{3} + a_3 \tfrac{t^4}{4}. \tag{8.33}$$

It is worth noting that at $t = 0$, $N(0,f) = N_0(f)$ and

$$\left.\frac{dN}{dt}\right|_{t=0} = a_0 = \alpha_1[N_0^2(N_{+0} + N_{-0}) - 2N_0N_{+0}N_{-0}] = \alpha_{10}. \tag{8.34}$$

The integrals for $N_+(t,f)$ and $N_-(t,f)$ are simply

$$N_+(t,f) = N_{+0}(f_+) + \left(\tfrac{\alpha_2}{\alpha_1}\right)\left[a_0 t + a_1 \tfrac{t^2}{2} + a_2 \tfrac{t^3}{3} + a_3 \tfrac{t^4}{4}\right] \tag{8.35}$$

$$N_-(t,f) = N_{-0}(f_-) + \left(\tfrac{\alpha_3}{\alpha_1}\right)\left[a_0 t + a_1 \tfrac{t^2}{2} + a_2 \tfrac{t^3}{3} + a_3 \tfrac{t^4}{4}\right]. \tag{8.36}$$

Equations (8.33), (8.34), and (8.35) are the second approximate solutions. These are nonlinear.

8.4.3 Third approximation

In this approximation, we shall update the differential equations (8.16), (8.17), and (8.18) by new values of N, N_+, and N_- obtained in equations (8.33), (8.35), and (8.36). The differential equation (8.37) becomes

$$\begin{aligned}
\frac{dN}{dt} &= \alpha_1[N^2(N_+ + N_-) - 2NN_+N_-] \\
&= \alpha_1\Bigg[\ \{N_0^2(N_{+0} + N_{-0}) - 2N_0N_{+0}N_{-0}\} \\
&\quad + \Bigg\{\frac{N_0^2(\alpha_2 + \alpha_3)}{\alpha_1} + 2N_0(N_{+0} + N_{-0}) \\
&\quad - 2\left(N_{+0}N_{-0} + \left(\frac{N_0}{\alpha_1}\right)(\alpha_3 N_{+0} + \alpha_2 N_{-0})\right)\Bigg\} A \\
&\quad + \Bigg\{\frac{2N_0(\alpha_2 + \alpha_3)}{\alpha_1} + (N_{+0} + N_{-0}) \\
&\quad - 2\left(\frac{N_0\alpha_2\alpha_3}{\alpha_1^2} + \frac{N_{+0}\alpha_3 + N_{-0}\alpha_2}{\alpha_1}\right)\Bigg\} A^2 \\
&\quad + \left\{\frac{\alpha_2 + \alpha_3}{\alpha_1} - \frac{2\alpha_2\alpha_3}{\alpha_1^2}\right\} A^3\Bigg],
\end{aligned}$$

or we can write the above equation as

$$\frac{dN}{dt} = \alpha_1[\beta_0 + \beta_1 A + \beta_2 A^2 + \beta_3 A^3], \tag{8.37}$$

where

$$\left.\begin{aligned}
\beta_0 &= N_0^2(N_{+0} + N_{-0}) - 2N_0 N_{+0} N_{-0} \\
\beta_1 &= 2N_0(N_{+0} + N_{-0}) + N_0^2(\alpha_2 + \alpha_3) \\
&\quad - 2\left\{N_{+0}N_{-0} + \left(\tfrac{N_0}{\alpha_1}\right)(\alpha_3 N_{+0} + \alpha_2 N_{-0})\right\} \\
\beta_2 &= (N_{+0} + N_{-0}) + \frac{2N_0(\alpha_2 + \alpha_3)}{\alpha_1} \\
&\quad - 2\left\{\frac{N_0\alpha_2\alpha_3}{\alpha_1^2} + \frac{\alpha_3 N_{+0} + \alpha_2 N_{-0}}{\alpha_1}\right\} \\
\beta_3 &= \frac{\alpha_2 + \alpha_3}{\alpha_1} - \frac{2\alpha_2\alpha_3}{\alpha_1^2} \\
A &= a_0 t + a_1\tfrac{t^2}{2} + a_2\tfrac{t^3}{3} + a_3\tfrac{t^4}{4}.
\end{aligned}\right\} \tag{8.38}$$

Integrating equation (8.37) with respect to time t, from $t=0$ to $t=t$, we get

$$N(t,f) = N_0(f) + \alpha_1\left[\beta_0 t + \beta_1\int_0^t A\,dt + \beta_2\int_0^t A^2 dt + \beta_3\int_0^t A^3 dt\right], \tag{8.39}$$

we show the calculation as follows

$$\begin{aligned}
\int_0^t A\,dt &= \int_0^t \left(a_0 t + a_1\frac{t^2}{2} + a_2\frac{t^3}{3} + a_3\frac{t^4}{4}\right) dt \\
&= a_0\frac{t^2}{2} + a_1\frac{t^3}{6} + a_2\frac{t^4}{12} + a_3\frac{t^5}{20}
\end{aligned}$$

$$\begin{aligned}
\int_0^t A^2 dt &= \int_0^t \left(a_0 t + a_1\frac{t^2}{2} + a_2\frac{t^3}{3} + a_3\frac{t^4}{4}\right)^2 dt \\
&= \frac{a_0^2 t^3}{3} + \frac{a_0 a_1 t^4}{4} + \left(\frac{a_1^2}{20} + \frac{2a_0 a_2}{15}\right)t^5 \\
&\quad + \left(\frac{a_0 a_3}{12} + \frac{a_1 a_2}{18}\right)t^6 + \left(\frac{a_1 a_3}{28} + \frac{a_2^2}{63}\right)t^7 \\
&\quad \times \frac{a_2 a_3 t^8}{48} + \frac{a_3^2 t^9}{144}
\end{aligned}$$

$$
\begin{aligned}
\int_0^t A^3 dt &= \int_0^t \left(a_0 t + a_1 \frac{t^2}{2} + a_2 \frac{t^3}{3} + a_3 \frac{t^4}{4}\right)^3 dt \\
&= a_0^3 \frac{t^4}{4} + \frac{3a_0^2 a_1 t^5}{10} + \left(\frac{a_0 a_1^2}{8} + \frac{a_0^2 a_2}{6}\right) t^6 \\
&\quad + \left(\frac{a_1^3}{56} + \frac{3a_0^2 a_3}{28} + \frac{a_0 a_1 a_2}{7}\right) t^7 \\
&\quad + \left(\frac{3a_0 a_1 a_3}{32} + \frac{a_1^2 a_2}{32} + \frac{a_0 a_2^2}{24}\right) t^8 \\
&\quad + \left(\frac{a_0 a_2 a_3}{18} + \frac{a_1 a_2^2}{54} + \frac{a_1^2 a_3}{48}\right) t^9 \\
&\quad + \left(\frac{3a_0 a_3^2}{160} + \frac{a_1 a_2 a_3}{40} + \frac{a_2^3}{270}\right) t^{10} \\
&\quad + \left(\frac{3a_1 a_3^2}{352} + \frac{a_2^2 a_3}{132}\right) t^{11} + \frac{a_2 a_3^2}{192} t^{12} \\
&\quad + \frac{a_3^3}{832} t^{13}.
\end{aligned} \tag{8.40}
$$

Similarly, we can write the integrals for $N_+(t,f)$ and $N_-(t,f)$ in the following form

$$
N_+(t,f) = N_{+0}(f_+) + \alpha_1 \left[\beta_0 t + \beta_1 \int_0^t A dt + \beta_2 \int_0^t A^2 dt + \beta_3 \int_0^t A^3 dt\right] \tag{8.41}
$$

$$
N_-(t,f) = N_{-0}(f_-) + \alpha_1 \left[\beta_0 t + \beta_1 \int_0^t A dt + \beta_2 \int_0^t A^2 dt + \beta_3 \int_0^t A^3 dt\right]. \tag{8.42}
$$

Equations (8.39), (8.41), and (8.42) are the third approximate solutions and are highly nonlinear in t, a polynomial of degree thirteen. The parameters defined above are functions of N_0, N_{+0}, and N_{-0}, i.e. they are a function of frequency f.

8.5 Adomian decomposition method

We have the following relationship from equations (8.16), (8.17), and (8.18) as

$$
\frac{dN_+}{dt} = \left(\frac{\alpha_2}{\alpha_1}\right) \frac{dN}{dt} \tag{8.43}
$$

$$
\frac{dN_-}{dt} = \left(\frac{\alpha_3}{\alpha_1}\right) \frac{dN}{dt}. \tag{8.44}
$$

To determine the solution for N, integrating equations (8.43) and (8.44) with respect to time from $t=0$ to $t=t$, i.e.

$$\int_0^t \frac{dN_+}{dt}dt = \left(\frac{\alpha_2}{\alpha_1}\right)\int_0^t \frac{dN}{dt}dt \tag{8.45}$$

$$\int_0^t \frac{dN_-}{dt}dt = \left(\frac{\alpha_3}{\alpha_1}\right)\int_0^t \frac{dN}{dt}dt, \tag{8.46}$$

we obtain

$$N_+ = \frac{\alpha_2}{\alpha_1}N - C_1 \tag{8.47}$$

$$N_- = \frac{\alpha_3}{\alpha_1}N - C_2 \tag{8.48}$$

where

$$C_1 = \frac{\alpha_2}{\alpha_1}N_0 - N_{+0} \tag{8.49}$$

$$C_2 = \frac{\alpha_3}{\alpha_1}N_0 - N_{-0}. \tag{8.50}$$

Substituting equations (8.47) and (8.48) into equation (8.16), we get

$$\frac{dN}{dt} = AN^3(t,f) + BN^2(t,f) + CN(t,f), \tag{8.51}$$

where

$$A = (\alpha_2 + \alpha_3) - \frac{2\alpha_2\alpha_3}{\alpha_1} \tag{8.52}$$

$$B = 2\alpha_3 C_1 + 2\alpha_2 C_2 - \alpha_1 C_3, \qquad C_3 = C_1 + C_2 \tag{8.53}$$

$$C = -2\alpha_1 C_1 C_2. \tag{8.54}$$

Integrating equation (8.51) with respect to time from $t = 0$ to $t = t$, i.e.

$$\int_0^t \frac{dN}{dt}dt = A\int_0^t N^3(t,f)dt + B\int_0^t N^2(t,f)dt + C\int_0^t N(t,f)dt \tag{8.55}$$

$$\Rightarrow N(t,f) - N(0,f) = A\int_0^t N^3(t,f)dt + B\int_0^t N^2(t,f)dt + C\int_0^t N(t,f)dt$$

$$\Rightarrow N(t,f) = N(0,f) + A\int_0^t N^3(t,f)dt + B\int_0^t N^2(t,f)dt + C\int_0^t N(t,f)dt. \tag{8.56}$$

The above equation is in canonical form, where $N(0,f)$ is known. Let us expand the $N(t,f)$ in the following manner

$$N(t,f) = \sum_{n=0}^{\infty} N_n = N_0 + N_1 + N_2 + N_3 + \cdots . \tag{8.57}$$

Substituting equation (8.57) into equation (8.56) we get

$$\begin{aligned}
N_0 + N_1 + N_2 + N_3 + \cdots &= N(0,f) \\
&\quad + \left[A \int_0^t (N_0 + N_1 + N_2 + N_3 + \cdots)^3 dt \right. \\
&\quad + B \int_0^t (N_0 + N_1 + N_2 + N_3 + \cdots)^2 dt \\
&\quad \left. + C \int_0^t (N_0 + N_1 + N_2 + N_3 + \cdots) dt \right] \\
&= N(0,f) + A \left[\int_0^t N_0^3 dt + \int_0^t (3N_0^2 N_1 + 3N_1^2 N_0 \right. \\
&\quad + N_1^3) dt + \int_0^t (3N_0^2 N_2 + 3N_2^2 N_0 + 6N_0 N_1 N_2 \\
&\quad + 3N_1^2 N_2 + 3N_2^2 N_1 + N_2^3) dt + \int_0^t (3N_0^2 N_3 \\
&\quad + 6N_0 N_1 N_3 + 6N_0 N_2 N_3 + 3N_0 N_3^2 + 3N_1^2 N_3 \\
&\quad + 6N_1 N_2 N_3 + 3N_1 N_3^2 + 3N_2^2 N_3 + 3N_2 N_3^2 \\
&\quad \left. + \ N_3^3) dt + \cdots \right] + B \left[\int_0^t N_0^2 dt \right. \\
&\quad + \int_0^t (2N_0 N_1 + N_1^2) dt + \int_0^t (2N_0 N_2 + 2N_1 N_2 \\
&\quad + N_2^2) dt + \int_0^t (2N_0 N_3 + 2N_1 N_3 + 2N_2 N_3 \\
&\quad \left. + \ N_3^2) dt + \cdots \right] + C \left[\int_0^t N_0 dt + \int_0^t N_1 dt \right. \\
&\quad \left. + \int_0^t N_2 dt + \int_0^t N_3 dt + \cdots \right].
\end{aligned} \tag{8.58}$$

Comparing the terms of the left- and right-hand series sequentially, we have

$$N_0 = N(0,f) = \alpha_0. \tag{8.59}$$

From the above known term we can find N_1 as

$$\begin{aligned}
N_1 &= A \int_0^t N_0^3 dt + B \int_0^t N_0^2 dt + C \int_0^t N_0 dt \\
&= (AN_0^3 + BN_0^2 + CN_0)t \\
&= \alpha_1 t \qquad \text{where} \qquad \alpha_1 = AN_0^3 + BN_0^2 + CN_0.
\end{aligned} \tag{8.60}$$

Similarly,

$$
\begin{aligned}
N_2 &= A\int_0^t (3N_0^2N_1 + 3N_1^2N_0 + N_1^3)dt \\
&\quad + B\int_0^t (2N_0N_1 + N_1^2)dt + C\int_0^t N_1 dt \\
&= A\int_0^t (3\alpha_0^2\alpha_1 t + 3\alpha_1^2\alpha_0 t^2 + \alpha_1^3 t^3)dt \\
&\quad + B\int_0^t (2\alpha_0\alpha_1 t + \alpha_1^2 t^2)dt + C\int_0^t \alpha_1 t dt \\
&= A\left[\frac{3}{2}\alpha_0^2\alpha_1 t^2 + \alpha_1^2\alpha_0 t^3 + \frac{1}{4}\alpha_1^3 t^4\right] \\
&\quad + B\left[\alpha_0\alpha_1 t^2 + \frac{1}{3}\alpha_1^2 t^3\right] + C\left[\frac{1}{2}\alpha_1 t^2\right] \\
&= \left[\frac{1}{4}A\alpha_1^3\right]t^4 + \left[A\alpha_1^2\alpha_0 + \frac{1}{3}B\alpha_1^2\right]t^3 \\
&\quad + \left[\frac{3}{2}A\alpha_0^2\alpha_1 + B\alpha_0\alpha_1 + \frac{1}{2}C\alpha_1\right]t^2 \\
&= \xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2
\end{aligned}
\tag{8.61}
$$

$$\xi_1 = \frac{1}{4}A\alpha_1^3$$

$$\xi_2 = A\alpha_1^2\alpha_0 + \frac{1}{3}B\alpha_1^2$$

$$\xi_3 = \frac{3}{2}A\alpha_0^2\alpha_1 + B\alpha_0\alpha_1 + \frac{1}{2}C\alpha_1$$

$$
\begin{aligned}
N_3 &= A\int_0^t (3N_0^2N_2 + 3N_2^2N_0 + 6N_0N_1N_2 \\
&\quad + 3N_1^2N_2 + 3N_2^2N_1 + N_2^3)dt + B\int_0^t (2N_0N_2 \\
&\quad + 2N_1N_2 + N_2^2)dt + C\int_0^t N_2 dt \\
&= A\int_0^t \Big\{3\alpha_0^2(\xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2) + 3\alpha_0(\xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2)^2 \\
&\quad \times 6\alpha_0\alpha_1 t(\xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2) + 3\alpha_1^2 t^2(\xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2) \\
&\quad + 3\alpha_1 t(\xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2)^2 + (\xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2)^3\Big\} \\
&\quad + B\int_0^t \Big\{2\alpha_0(\xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2) + 2\alpha_1 t(\xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2) \\
&\quad + (\xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2)^2\Big\}dt + C\int_0^t (\xi_1 t^4 + \xi_2 t^3 + \xi_3 t^2)dt.
\end{aligned}
\tag{8.62}
$$

Expanding all the terms and arranging like terms we get

$$
\begin{aligned}
N_3 = {} & \tfrac{1}{13}A\xi_1^3 t^{13} + \tfrac{1}{4}A\xi_1^2\xi_2 t^{12} + \tfrac{3}{11}A(\xi_1\xi_2^2 + \xi_1^2\xi_3)t^{11} \\
& + \left\{\tfrac{3}{10}A\alpha_1\xi_1^2 + \tfrac{1}{10}A(6\xi_1\xi_2\xi_3 + \xi_2^3)\right\} t^{10} \\
& + \left\{\tfrac{1}{3}A\alpha_0\xi_1^2 + \tfrac{2}{3}A\alpha_1\xi_1\xi_2 + \tfrac{1}{3}A(\xi_1\xi_3^2 + \xi_2^2\xi_3)\right. \\
& \left. + \tfrac{1}{9}B\xi_1^2\right\} t^9 + \left\{\tfrac{3}{4}A\alpha_0\xi_1\xi_2 + \tfrac{3}{8}A\alpha_1(2\xi_1\xi_3 + \xi_2^2)\right. \\
& \left. + \tfrac{3}{8}A\xi_2\xi_3^2 + \tfrac{1}{4}B\xi_1\xi_2\right\} t^8 + \left\{\tfrac{3}{7}A\alpha_0(2\xi_1\xi_3 + \xi_2^2)\right. \\
& + \tfrac{3}{7}A\alpha_1^2\xi_1 + \tfrac{6}{7}A\alpha_1\xi_2\xi_3 + \tfrac{1}{7}A\xi_3^3 \\
& \left. + \tfrac{1}{7}B(2\xi_1\xi_3 + \xi_2^2)\right\} t^7 + \left\{A\alpha_0\xi_2\xi_3 + A\alpha_0\alpha_1\xi_1\right. \\
& \left. + \tfrac{1}{2}A(\alpha_1^2\xi_2 + \alpha_1\xi_3^2) + \tfrac{1}{3}B(\alpha_1\xi_1 + \xi_2\xi_3)\right\} t^6 \\
& + \left\{\tfrac{3}{5}A(\alpha_0^2\xi_1 + \alpha_0\xi_3^2 + \alpha_1^2\xi_3) + \tfrac{6}{5}A\alpha_0\alpha_1\xi_2\right. \\
& \left. + \tfrac{2}{5}B(\alpha_0\xi_1 + \alpha_1\xi_2) + \tfrac{1}{5}(B\xi_3^2 + C\xi_1)\right\} t^5 \\
& + \left\{\tfrac{3}{4}A\alpha_0^2\xi_2 + \tfrac{3}{2}A\alpha_0\alpha_1\xi_3 + \tfrac{1}{2}B(\alpha_0\xi_2\right. \\
& \left. + \alpha_1\xi_3) + \tfrac{C}{4}\xi_2\right\} t^4 + \left\{A\alpha_0^2\xi_3 + \tfrac{2}{3}B\alpha_0\xi_3 + \tfrac{1}{3}C\xi_3\right\} t^3.
\end{aligned}
\tag{8.63}
$$

Thus, the approximate solution for N will be the sum of the above four terms and is highly nonlinear in t

$$N = N_0 + N_1 + N_2 + N_3. \tag{8.64}$$

8.6 Fourth-order Runge–Kutta method

In this section, we shall try to determine the solutions of equations (8.16), (8.17), and (8.18) with their initial conditions [equation (8.20)]. The numerical scheme that is required to solve the initial value problem will be discussed very briefly. For a given set of initial conditions, we will try to solve these highly nonlinear first-order ordinary differential equations. The scheme is as follows: Rewriting equations (8.16), (8.17), and (8.18) in the functional form, we have

$$\frac{dN}{dt} = f(t, N, N_+, N_-) \tag{8.65}$$

$$\frac{dN_+}{dt} = g(t, N, N_+, N_-) \tag{8.66}$$

$$\frac{dN_-}{dt} = h(t, N, N_+, N_-), \tag{8.67}$$

where

$$f = \alpha_1[N^2(N_+ + N_-) - 2NN_+N_-] \tag{8.68}$$

$$g = \alpha_2[N^2(N_+ + N_-) - 2NN_+N_-] \tag{8.69}$$

$$h = \alpha_3[N^2(N_+ + N_-) - 2NN_+N_-]. \tag{8.70}$$

The fourth-order Runge–Kutta integration scheme implies the solution of the $(j+1)$th time step as

$$N^{j+1} = N^j + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{8.71}$$

$$N_+^{j+1} = N^j + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \tag{8.72}$$

$$N_-^{j+1} = N^j + \tfrac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4), \tag{8.73}$$

where

$$\left.\begin{aligned} k_1 &= (\Delta t)f(t, N, N_+, N_-) \\ l_1 &= (\Delta t)g(t, N, N_+, N_-) \\ m_1 &= (\Delta t)h(t, N, N_+, N_-) \end{aligned}\right\} \tag{8.74}$$

$$\left.\begin{aligned} k_2 &= (\Delta t)f\left(t + \tfrac{\Delta t}{2}, N + \tfrac{k_1}{2}, N_+ + \tfrac{l_1}{2}, N_- + \tfrac{m_1}{2}\right) \\ l_2 &= (\Delta t)g\left(t + \tfrac{\Delta t}{2}, N + \tfrac{k_1}{2}, N_+ + \tfrac{l_1}{2}, N_- + \tfrac{m_1}{2}\right) \\ m_2 &= (\Delta t)h\left(t + \tfrac{\Delta t}{2}, N + \tfrac{k_1}{2}, N_+ + \tfrac{l_1}{2}, N_- + \tfrac{m_1}{2}\right) \end{aligned}\right\} \tag{8.75}$$

$$\left.\begin{aligned} k_3 &= (\Delta t)f\left(t + \tfrac{\Delta t}{2}, N + \tfrac{k_2}{2}, N_+ + \tfrac{l_2}{2}, N_- + \tfrac{m_2}{2}\right) \\ l_3 &= (\Delta t)g\left(t + \tfrac{\Delta t}{2}, N + \tfrac{k_2}{2}, N_+ + \tfrac{l_2}{2}, N_- + \tfrac{m_2}{2}\right) \\ m_3 &= (\Delta t)h\left(t + \tfrac{\Delta t}{2}, N + \tfrac{k_2}{2}, N_+ + \tfrac{l_2}{2}, N_- + \tfrac{m_2}{2}\right) \end{aligned}\right\} \tag{8.76}$$

$$\left.\begin{aligned} k_4 &= (\Delta t)f\,(t+\Delta t, N+k_3, N_+ + l_3, N_- + m_3) \\ l_4 &= (\Delta t)f\,(t+\Delta t, N+k_3, N_+ + l_3, N_- + m_3) \\ m_4 &= (\Delta t)f\,(t+\Delta t, N+k_3, N_+ + l_3, N_- + m_3) \end{aligned}\right\}. \tag{8.77}$$

Equations (8.71) to (8.73) specify the action transfer in the air–sea momentum exchange. Once the values of N, N_+ and N_- at the $(j+1)$th time step have been determined, the time derivative $(\frac{dN}{dt})$ can be obtained from the equation

$$\left(\frac{dN}{dt}\right)^{j+1} = \alpha_1[N^2(N_+ + N_-) - 2NN_+N_-]^{j+1}. \tag{8.78}$$

We shall carry out this numerical integration for the range from $t=0$ to $t=2000$ s.

8.7 Results and discussion

We shall discuss our results in graphical form and compare the graphs obtained from the numerical method with the analytical one. The initial conditions used in these calculations are as follows: At $t=0$, we use the JONSWAP spectrum as the initial condition. The expression for this spectrum is given below (see Rahman [15, 17])

$$N(f) = \alpha g^2 \frac{f^{-5}}{(2\pi)^4} \exp\left\{\frac{-5}{4}\left(\frac{f}{f_p}\right)^{-4}\right\} \gamma^{\exp\left\{-\frac{(f-f_p)^2}{2\tau^2 f_p^2}\right\}}, \tag{8.79}$$

where $\alpha=0.01$, $\gamma=3.3$, $\tau=0.08$, and $f_p=0.3$. Here, f_p is called the peak frequency of the JONSWAP spectrum. Similarly, the initial conditions for N_+ N_- are the following:

$$N_+(f_+) = N((1+\lambda)f) \tag{8.80}$$

$$N_-(f_-) = N((1-\lambda)f). \tag{8.81}$$

The corresponding spreading of the directional spectrum (f, θ) was found to depend primarily on $\frac{f}{f_p}$

$$N(f,\theta) = \tfrac{\beta}{2} N(f)\operatorname{sech}^2\beta(\theta - \overline{\theta}(f)), \tag{8.82}$$

where $\overline{\theta}$ is the mean direction

$$\beta = \begin{cases} 2.61\left(\dfrac{f}{f_p}\right)^{1.3} & \text{for } 0.56 < \dfrac{f}{f_p} < 0.95 \\ 2.28\left(\dfrac{f}{f_p}\right)^{-1.3} & \text{for } 0.95 < \dfrac{f}{f_p} < 1.6 \\ 1.24 & \text{otherwise.} \end{cases} \tag{8.83}$$

The parameters involved in this problem for the one-dimensional deep ocean case are given by

$$\begin{aligned}\alpha_1 &= -2Cg^{-8}f^{19}\Delta k\\ \alpha_2 &= Cg^{-8}f^{19}\Delta k_+\\ \alpha_3 &= Cg^{-8}f^{19}\Delta k_-\\ C &= 3\times 10^7\\ g &= 9.8\,\mathrm{m}/s^2,\end{aligned}$$

in which

$$\begin{aligned}\Delta k &= \frac{8\pi^2 f\Delta f}{g}\\ \Delta k_+ &= \frac{8\pi^2 f(1+\lambda)^2\Delta f}{g}\\ \Delta k_- &= \frac{8\pi^2 f(1-\lambda)^2\Delta f}{g}.\end{aligned}$$

The frequency range is taken as $f=0.0\,\mathrm{Hz}$ to $f=2\,\mathrm{Hz}$ with step size of $\Delta f=0.001$. The graphical results of the four methods by taking all the parameters stated above are presented in Figure 8.1. Figures 8.2 and 8.3 show the plots between $\gamma=2$ and $\gamma=4$, respectively, keeping other parameters constant.

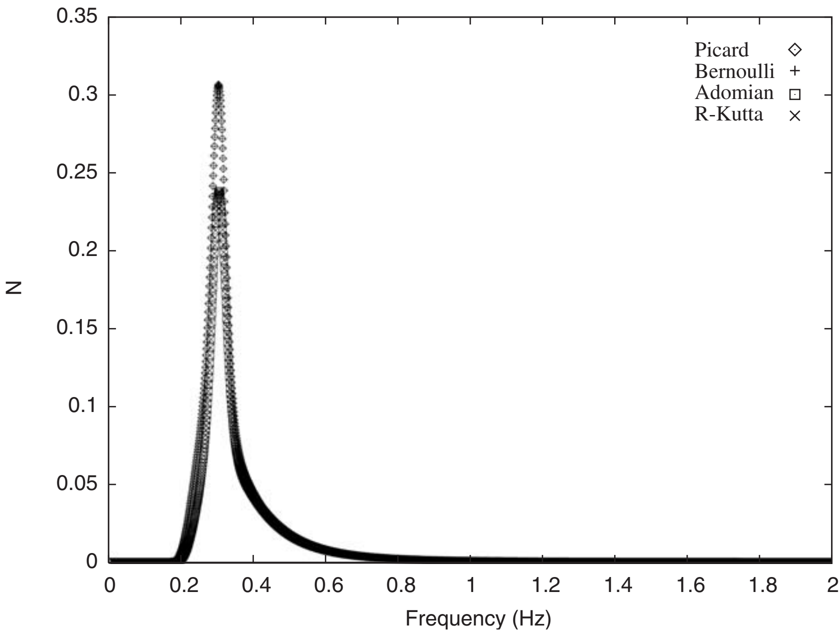

Figure 8.1: Nonlinear transfer action function $N(f)$ versus the frequency f using all the parameters stated above.

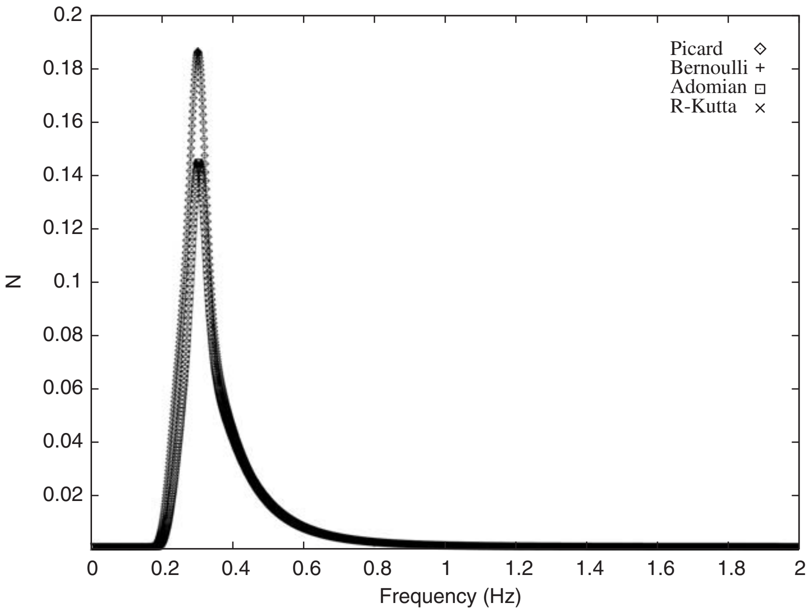

Figure 8.2: Nonlinear transfer action function $N(f)$ versus the frequency f using $\gamma = 2$ keeping other parameters constant.

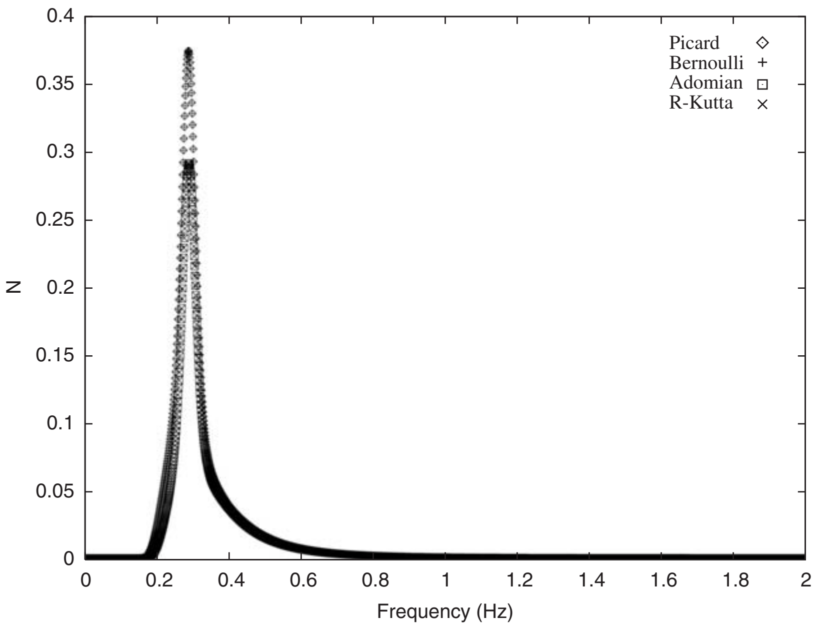

Figure 8.3: Nonlinear transfer action function $N(f)$ versus the frequency f using $\gamma = 4$ keeping other parameters constant.

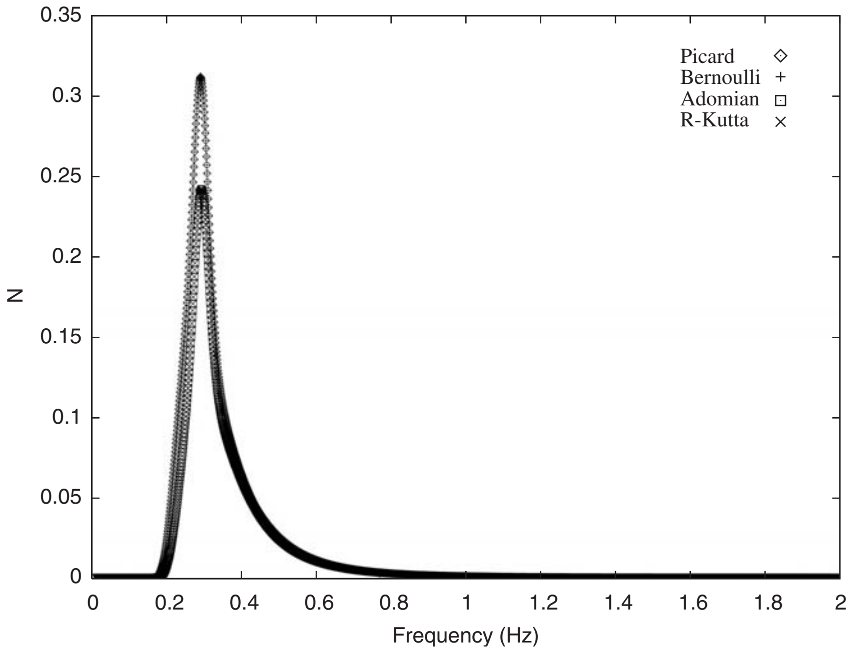

Figure 8.4: Nonlinear transfer action function $N(f)$ versus the frequency f using $C = 3 \times 10^2$ keeping other parameters constant.

The plots in Figure 8.4 are obtained at $C = 3 \times 10^2$, where for each case time is taken as 2000 s. All the results show excellent agreement. To determine the net total action transfer, N_{total}, we use the following formula:

$$N_{\text{total}} = \int_{\theta=0}^{2\pi} \int_{f=0}^{\infty} N(f, \theta)\, df\, d\theta. \tag{8.84}$$

But in a practical situation, the limit of the infinite integral takes the finite values

$$N_{\text{total}} = \int_{\theta=0}^{2\pi} \int_{f=0}^{2} N(f, \theta)\, df\, d\theta. \tag{8.85}$$

The upper limit of the f-integral is assumed to be 2 Hz, which seems to be a realistic cutoff frequency instead of infinity, with the understanding that the contribution to the integral from 2 to infinity is insignificant. The $N(f)$ at a certain time for different frequencies is highly nonlinear. Thus, to plot N_{total} in the time scale, we first obtain the values at different time scale and then graph these values against time. From the analytical solution for $N(f)$ we can find the values for N_{total} using either Gaussian quadrature or the $\frac{3}{8}$ Simpson's rule of integration. This analysis is valid only for deep-water ocean.

8.8 Green's function method for waves

The application of Green's function in calculation of flow characteristics around a submerged sphere in regular waves is presented in this section. We assume that the fluid is homogeneous, inviscid, and incompressible, and the fluid motion is irrotational and small. Two methods based on the boundary integral equation method (BIEM) are applied to solve the associated problems. The first is the *flat panel method* (FPM) using triangular flat patches to model the body and the second is using the modified form of the Green's function in order to make it nonsingular and amenable to apply directly the Gaussian quadrature formulas.

8.8.1 Introduction

A combination of two independent classical problems should be considered in order to find the hydrodynamic characteristics of motion of a body in time harmonic waves. One is the radiation problem where the body undergoes prescribed oscillatory motions in otherwise calm fluid, and the other is the diffraction problem where the body is held fixed in the incident wave field and determines the influence of it over the incident wave. These boundary value problems can be formulated as two different types of integral equations. The so-called direct boundary integral formulation function as a superposition of a single-layer and double-layer potentials. Another is referred to as the indirect boundary integral formulation, which represents an unknown function with the aid of a source distribution of Green's function with fictitious singularities of adjustable strength, Yang [25].

One of the most widely used BIEM is that of Hess and Smith [7], in which the indirect method is used to solve the problem of potential flow without the free surface effect. Hess and Smith [7] subdivided the body surface into n quadrilateral flat panels over which the source strength distribution was assumed to be uniform. Webster [23] developed a method that can be regarded as an extension of Hess and Smith's method by using triangular patches over which the source strength distribution is chosen to vary linearly across the patch and the panels are submerged somewhat below the actual surface of the body. A modification of the Hess and Smith [7] method that is devised by Landweber and Macagno [11]. The method mainly differs in the treatment of the singularity of the kernel of the integral equations and applying the Gaussian quadrature to obtain numerical solutions.

The application of Green's function in calculating the flow characteristics around a submerged sphere in regular waves is presented in this section. It is assumed that the fluid is homogeneous, inviscid, and incompressible and the flow is irrotational. Such a boundary value problem can be recast as integral equations via Green's theorem. Therefore, the associated problems with the motion of submerged or floating bodies in regular waves can be solved with BIEM. There are two approaches in solving such a problem with BIEM: the Green function method (GFM) and the Rankine source method (RSM). In the RSM, the fundamental solution is applied and the whole flow boundaries (the body surface, the free surface, and the bottom

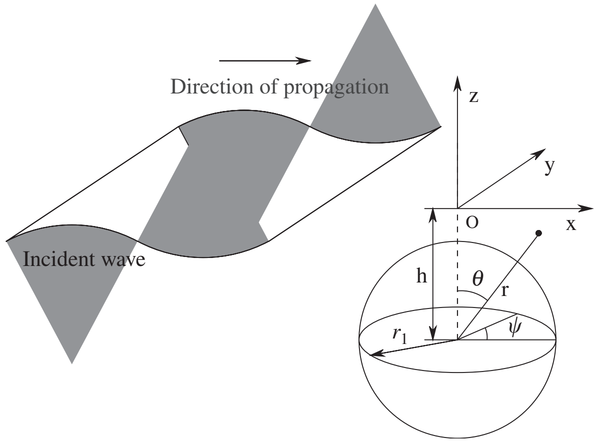

Figure 8.5: Sketch of the problem geometry and coordinates definition.

surface) are discretized and the boundary conditions are imposed to find the solution. It has the advantages of: the influence matrix can be evaluated easily and it can be extended to a nonlinear free surface conditions. In the GFM, a function is found by considering all the boundary conditions except the body surface boundary condition and then by imposing the body boundary condition to the discretize body surface, the associated integral equations are solved numerically to find the velocity potentials and the flow characteristics. The advantages of GFM in comparison with the RSM are that the integral equations should only be solved around the surface of the body and no restriction applied to the free surface to implement the radiation condition.

The direct boundary integral formulation along with the GFM is applied to find the hydrodynamic characteristics of the motion of a sphere in time harmonic waves by two different methods. In the first method, the original form of the free-surface Green's function is used to find the solutions by modelling the body as a faceted form of triangular patches. In the second method, the nonsingular form of the free surface Green's function along with the Gauss–Legendre quadrature formula are applied to find the solutions for the associated problems.

8.8.2 Mathematical formulation

Two sets of coordinate systems were considered (see Figure 8.5). One is a right-handed coordinate system fixed in the fluid with oz opposing the direction of gravity and oxy lying in the undisturbed free surface. The other set is the spherical coordinate system (r, θ, ψ) with the origin at the centre of sphere. The total velocity potential may be written as

$$\Phi(x,y,z,t) = \Re[\phi(x,y,z)e^{-i\omega t}], \tag{8.86}$$

where ϕ is the time-independent velocity potential. It can be decomposed in the general case as

$$\phi(x,y,z) = \sum_{j=1}^{6} \eta_j \phi_j(x,y,z) + A(\phi_I + \phi_D), \tag{8.87}$$

where η_j is the amplitude for each of the six degrees of freedom of the body and ϕ_j is the time-independent velocity potential corresponding to each mode of oscillation of the body of unit amplitude. Due to the geometrical symmetry of the sphere, there are only three modes of oscillations responding to disturbance from any given direction. These three modes of motion are surge, heave, and pitch ($j = 1, 3, 5$) considering the incident wave propagates along the x-axis. ϕ_I is the spatial incoming wave velocity potential with unit amplitude and ϕ_D is the spatial wave diffraction velocity potential of unit amplitude. Based on the assumption of linearized theory, the complex spatial part of the velocity potentials, ϕ, must satisfy

$$\nabla^2 \phi = 0 \quad for \quad r \geq r_1, \quad z \leq 0 \tag{8.88}$$

$$K\phi - \frac{\partial \phi}{\partial z} = 0 \quad at\, z = 0, \tag{8.89}$$

$$\partial \phi / \partial z = 0 \quad for \quad z \to -\infty \tag{8.90}$$

$$\sqrt{R}\left\{\frac{\partial}{\partial R} - iK\right\}\phi = 0 \quad for \quad R \to \infty \tag{8.91}$$

$$\frac{\partial \phi_j}{\partial n} = -i\omega n_j \quad j = 1, 3, 5 \tag{8.92}$$

$$\left(\frac{\partial}{\partial \mathbf{n}}\right)(\phi_I + \phi_D) = 0, \tag{8.93}$$

where $K = \omega^2/g$ is called the wavenumber, $R = \sqrt{x^2 + y^2}$, $\mathbf{n}$ the outward unit normal vector of the body surface to the fluid , $n_1 = \sin\theta\cos\psi$, $n_3 = \cos\theta$, and $n_5 = 0$. Equations (8.92) and (8.93) are the kinematic boundary conditions on the body surface of the sphere for the radiation and diffraction problems, respectively.

The other boundary conditions on the body are the equations of motion of the body. It is supposed that the sphere is hydrostatically stable. This means that the centre of mass of the sphere should be under its centre of volume. The forces and moments acting on the body are the gravity force and the reaction of the fluid. The forces and moments of the fluid on the body can be determined from the following formulas:

$$\left\{\begin{matrix} \mathcal{F} \\ \mathcal{M} \end{matrix}\right\} = \int\int_S P \left\{\begin{matrix} \mathbf{n} \\ \mathbf{r} \times \mathbf{n} \end{matrix}\right\} dS. \tag{8.94}$$

The dynamic pressure P can be determined by applying the linearized Bernoulli's equation. It may be written as

$$P(x,y,z,t) = \Re[p(x,y,z)e^{-i\omega t}] = -\rho\frac{\partial \Phi}{\partial t}, \tag{8.95}$$

where ρ is the fluid density and the spatial pressure, p, can be related to the spatial velocity potential with $p = i\rho\omega\phi$. We can express the equation of motion of a body in waves by using indicial notations as

$$m_{ij}\ddot{a}_j = -c_{ij}a_j - \rho\int\int_S \Phi_t n_i \, dS, \tag{8.96}$$

where m_{ij} are the mass matrix coefficients, a_j the linear or angular displacements of the body, $a_j = \Re[\eta_j(x,y,z)e^{-i\omega t}]$, and c_{ij} the restoring force coefficient. Considering equation (8.87), the Φ_t may be written as:

$$\Phi_t(x,y,z,t) = \Re\left\{-i\omega\left[\sum_{j=1}^{6}\eta_j\phi_j(x,y,z) + A(\phi_I + \phi_D)\right]e^{-i\omega t}\right\}. \tag{8.97}$$

Taking into account equation (8.92), the integral in equation (8.96) for a component of radiation velocity potential Φ_j becomes

$$I_{ij} = \rho\int\int_S \Phi_{j_t} n_i dS = \rho\Re\left[\eta_j e^{-i\omega t}\int\int_S \phi_j\frac{\partial\phi_i}{\partial n}\,dS\right]. \tag{8.98}$$

The component of force and moments for radiation problem can be written in the form of

$$F_i = \Re\left\{\sum_{j=1}^{6}\eta_j e^{-i\omega t} f_{ij}\right\}, \tag{8.99}$$

where f_{ij} is a complex force coefficient,

$$f_{kj} = \omega^2\alpha_{kj} - i\omega\beta_{kj} = -i\omega\rho\int\int_{S_0}\phi_j\frac{\partial\phi_i}{\partial n}\,dS. \tag{8.100}$$

α_{kj} is the added mass coefficient and β_{kj} the damping coefficient. For the sphere, all the complex force coefficients are vanished due to the geometrical symmetry except for the surge and heave motions.

The exciting forces and moments on a body can be obtained by integration of the hydrodynamic pressure associated with the incident and diffraction velocity potentials over its surface. If we consider $\Phi_{ID} = \Phi_I + \Phi_D$, and substitute it in the last term of equation (8.96), the exciting forces and moments are obtained,

$$\begin{Bmatrix} F_{ei} \\ M_{ei} \end{Bmatrix} = \Re\left[-i\omega\rho\int\int_S (\phi_I + \phi_D)|_S \begin{Bmatrix} n_i \\ r\times n_i \end{Bmatrix} dS\right]e^{-i\omega t}. \tag{8.101}$$

Considering that $F_{ei} = \Re[f_{ei}e^{-i\omega t}]$ and equation (8.92), the complex exciting force f_{ei} can be calculated by

$$f_{ei} = \rho \int\int_S \phi_{ID}|_S \frac{\partial \phi_i}{\partial n}\, dS. \tag{8.102}$$

Taking into account equations (8.96), (8.99), and (8.101), the equations of motion for a submerged sphere in a time harmonic wave are

$$\begin{cases} -\omega^2(M+\alpha_{11})\eta_1 - i\omega\beta_{11}\eta_1 - \omega^2 MZ_g\eta_5 = f_{ex} & \text{for surge motion} \\ -\omega^2(M+\alpha_{33})\eta_3 + i\omega\beta_{33}\eta_3 = f_{ez} & \text{for heave motion} \\ -\omega^2 MZ_g\eta_1 - \omega^2 Mk^2\eta_5 + MgZ_g\eta_5 = 0 & \text{for pitch motion ,} \end{cases} \tag{8.103}$$

where η_1, η_3, and η_5 are the surge, heave, and pitch amplitudes, M is the mass of the sphere, Z_g is the position of the centre of gravity with respect to the centre of the sphere, α_{11} and α_{33} are added mass coefficients for surge and heave, β_{11} and β_{33} are the damping coefficients, f_{ex} and f_{ez} are the complex exciting forces and k is the radius of gyration. The linear motions along the x direction (surge) and rotational motion about y-axis (pitch) are coupled with each other but they are not coupled with the linear motion along z-axis (heave) of the sphere.

8.8.3 Integral equations

The radiation and diffraction problems are subjected to the Laplace equation in the fluid domain, linearized free-surface boundary condition, bottom condition indicates that there is no flux through the bottom of the fluid, radiation condition at infinity, and the Neumann condition at the mean position of the body. The potential for a unit source at $q(\xi, \eta, \zeta)$ defines the Green function G. The Green function with its first and second derivatives is continuous everywhere except at the point q. It can be interpreted as the response of a system at a field point $p(x, y, z)$ due to a delta function input at the source point $q(\xi, \eta, \zeta)$. This solution can be applied with the Green second theorem to derive the integral equation for the velocity potentials on the surface of the body,

$$2\pi\phi(p) + \int\int_{S_B} \phi(q)\frac{\partial G(p,q)}{\partial n(q)}\, dS = \int\int_{S_B} G(p,q)\frac{\partial \phi(q)}{\partial n(q)}\, dS. \tag{8.104}$$

The free surface Green function that satisfies all boundary conditions except the body boundary conditions is defined, Wehausen and Laitone [24], as

$$\begin{aligned} G = \frac{1}{r} + \frac{1}{r'} + 2K\, P.V. \int_0^\infty \frac{1}{k-K} e^{k(z+\xi)} J_0(kR)dk \\ - 2\pi i k e^{K(z+\xi)} J_0(KR), \end{aligned} \tag{8.105}$$

where

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

$$r' = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}$$

$$R = \sqrt{(x-\xi)^2 + (y-\eta)^2}. \tag{8.106}$$

In the radiation problem, the velocity potential for each mode of motion of the body can be obtained by solving the integral, equation (8.104) and imposing the body surface boundary condition, i.e. equation (8.92).

In the diffraction problem, the scattering velocity potential, ϕ_D, can be calculated with the integral equation (8.104), and then adding the incident velocity potential to find the diffraction velocity potential, ϕ_{ID}. The integral equation for ϕ_D can be written

$$2\pi\phi_D(p) + \int\int_{S_B} \phi_D(q)\frac{\partial G(p,q)}{\partial n(q)}\, dS = \int\int_{S_B} G(p,q)\frac{\partial \phi_D(q)}{\partial n(q)}\, dS. \tag{8.107}$$

If the second Green's theorem applied to the ϕ_I on the interior of the surface S_B, the integral equation is in the form of

$$-2\pi\phi_I(p) + \int\int_{S_B} \phi_I(q)\frac{\partial G(p,q)}{\partial n(q)}\, dS = \int\int_{S_B} G(p,q)\frac{\partial \phi_I(q)}{\partial n(q)}\, dS. \tag{8.108}$$

Combining equations (8.107) and (8.108) and considering the body surface condition for the diffraction problem, equation (8.93), it follows that

$$4\pi\phi_{ID}(p) + \int\int_{S_B} \phi(q)\frac{\partial G(p,q)}{\partial n(q)} dS = 4\pi\phi_I(p). \tag{8.109}$$

Flat panel method

The integral equations (8.104) and (8.109) are discretized by subdividing the surface of the sphere, S_B, into N triangular panels, S_b. It is assumed that the distribution of the velocity potential and its normal derivative are constant on each panel and equal to their values at the panel centroid. The discretized formula for each component of the radiation velocity potentials is

$$2\pi\phi_i + \sum_{j=1}^{N} \phi_j \int\int_{S_b} \frac{\partial G}{\partial n_q}\, dS_b = \sum_{j=1}^{N} \frac{\partial \phi_j}{\partial n_q} \int\int_{S_b} G\, dS_b, \tag{8.110}$$

and for the diffraction velocity potential is

$$4\pi\phi_{ID_i} + \sum_{j=1}^{N} \phi_{ID_j} \int\int_{S_b} \frac{\partial G}{\partial n_q} ds_b = 4\pi\phi_{I_i}. \tag{8.111}$$

The results are systems of N complex linear algebraic equations with N unknowns. It is necessary to deal with a global coordinate system $O_1 - x_1y_1z_1$ fixed in the body and a set of local coordinate systems, $\overline{O} - \overline{xyz}$ for each triangular patch. It is convenient to assume that the triangular panel lies in the $\bar{x}, \bar{y}$ *coordinate* plane. The system of linear algebraic equations may be presented in matrix form as

$$[A]\{x\} = \{B\}, \tag{8.112}$$

where $[A]$ is the coefficient matrix formed by the integration of the derivative of Green's function along the surface of each panel. $\{B\}$ is a vector formed by multiplication of the integration of the Green's function along the surface of each panel and the body surface boundary condition for the radiation problem. For the diffraction problem its elements are $B_i = 4\pi\phi_{I_i}$. $\{x\}$ is the unknown vector of velocity potentials around the body. The main tasks are

a) finding the elements $[A]$ and $\{B\}$,
b) solving the system of complex linear algebraic equations.

The Green's function consists of Rankine singularities, its image, and wave parts due to the free surface condition and the radiation condition. The parts of A_{ij} and B_{ij} concerned with the Rankine singularity and its image may be computed by transferring the surface integral to the line integral using Green's lemma. These parts of the A_{ij} and B_{ij} due to the wave are calculated for the centroid of the panel and assumed to be constant on each panel. For the accurate evaluation of the wave part of the Green's function and its gradient, see Refs. [3], [13], [15] and [21].

Nonsingular method

The Green's function may be written in the form of $G(p,q) = \frac{1}{r} + H(p,q)$, where the term $\frac{1}{r}$ is the singular part and $H(p,q)$ is the harmonic part of it. The type of singularity of the $\frac{1}{r}$ and its derivative are of the weak singularity. This type of singularity can be removed by adding and subtracting a proper function from the integrand so that the kernel becomes nonsingular. The method proposed by Landweber and Macagno [11] for the calculation of the potential flow about ship forms are presented as such an idea. The treatment of singularity in such a way will result in a regularized formulation of BIEM. These nonsingular forms of integral equations can be solved numerically by applying the standard quadrature formulas directly without the conventional boundary element approximation. It can be written by introducing the Gauss's flux theorem that

$$\int\int_{S_B} \frac{\partial}{\partial n_q} \left(\frac{1}{r}\right) dS_q = -2\pi, \tag{8.113}$$

and defined such a source distribution, $\sigma(q)$, that make the body equipotential of potential ϕ_e.

$$\int_{S_B} \sigma(q)\frac{1}{r}\, dS_q = \phi_e. \tag{8.114}$$

Considering equations (8.113) and (8.114), the integral equations yield the form of

$$\begin{aligned}
4\pi\phi_j(p) &+ \int\int_{S_B} [\phi_j(q) - \phi_j(p)]\frac{\partial}{\partial n_q}\left(\frac{1}{r}\right) dS_q + \int\int_{S_B} \phi_j(q)\frac{\partial H(p,q)}{\partial n_q}\, dS_q \\
&= \int\int_{S_B}\left[\frac{\partial \phi_j(q)}{\partial n_q} - \frac{\partial \phi_j(p)}{\partial n_p}\frac{\sigma(q)}{\sigma(p)}\right]\frac{1}{r} dS_q \\
&+ \int\int_{S_B} H(p,q)\frac{\partial \phi_j(q)}{\partial n_q} dS_q - \frac{\phi_e}{\sigma(p)}\frac{\partial \phi_j(p)}{\partial n_p}
\end{aligned} \tag{8.115}$$

$$\begin{aligned}
4\pi\phi_{ID}(p) &+ \int\int_{S_B} [\phi_{ID}(q) - \phi_{ID}(p)]\frac{\partial}{\partial n_q}\left(\frac{1}{r}\right) dS_q \\
&+ \int\int_{S_B} \phi_{ID}(q)\frac{\partial H(p,q)}{\partial n_q} dS_q = 4\pi\phi_I(p).
\end{aligned} \tag{8.116}$$

The source distribution, $\sigma(q)$ can be calculated through the iterative formula

$$\sigma_p^{k+1} = \sigma_p^k + \int\int_{S_B}\left(\sigma_q^k \frac{\partial}{\partial n_p}\left(\frac{1}{r}\right) - \sigma_q^k \frac{\partial}{\partial n_q}\left(\frac{1}{r}\right)\right) dS_q, \tag{8.117}$$

and since ϕ_e is constant in the interior of an equipotential surface, its value may conveniently be computed by locating point p at the origin. For more explanation of the method, see Ref. [9],

$$\phi_e = \int\int_{S_B} \frac{\sigma_q}{(x^2+y^2+z^2)^{1/2}} dS_q \tag{8.118}$$

The solution of the radiation and the diffraction velocity potentials with this method are obtained by discretizing the integral equations (8.115) and (8.116) and the relations (8.117) and (8.118) by applying the Gaussian quadrature. The source distribution, $\sigma(q)$, to make the body equipotential and ϕ_e may be calculated through the discretized formulas

$$\sigma_i^{(k+1)} = \sigma_i^{(k)} - \frac{1}{2\pi}\sum_{j=1, j\neq i}^{N}\left(\sigma_j^{(k)}\frac{r_{ij}.n_i}{r_{ij}^3} + \sigma_i^{(k)}\frac{r_{ij}.n_j}{r_{ij}^3}\right)\Delta s_j \omega_j$$

$$\phi_e = \sum_{j=1}^{N} \sigma_j (x_j^2 + y_j^2 + z_j^2)^{-\frac{1}{2}} \Delta s_j \omega_j, \tag{8.119}$$

where Δs_j is a scale factor with respect to the integration parameters, ω_j the weighting function for the Gauss–Legendre formula, and N the number of Gaussian quadrature points. The discretized equations for the radiation and diffraction problems are linear systems with N unknowns of velocity potentials. They can be represented in the matrix form as equation (8.112). The coefficient matrix and also the body surface boundary conditions are dependent on the geometry of the body. If the exact expression for the geometry can be used in the numerical computation the accuracy of the results will be highly improved. This is one of the most important features, using the exact geometry, of the nonsingular method in treating the integral equations. The elements of the coefficient matrix, $[A]$, are

$$A_{ii} = 4\pi + \sum_{j=1, j\neq i}^{N} \frac{\mathbf{r}_{ij}\cdot\mathbf{n}_j}{r_{ij}^3}\Delta s_j\omega_j + \frac{\partial H_{ii}}{\partial n_i}\Delta s_i\omega_i$$

$$A_{ij} = \left(-\frac{\mathbf{r}_{ij}.\mathbf{n}_j}{r_{ij}^3} + \frac{\partial H_{ij}}{\partial n_j}\right)\Delta s_j\omega_j. \tag{8.120}$$

The elements of $\{B\}$ for the diffraction problem are $B_i = 4\pi\phi_{I_i}$. For the radiation problem,

$$\{B\} = [C]\left\{\frac{\partial\phi}{\partial n}\right\}, \tag{8.121}$$

where the elements of the matrix $[C]$ are

$$C_{ii} = H_{ii}\Delta s_i\omega_i - \frac{\phi_e}{\sigma_i} - \sum_{j=1, j\neq i}^{N} \frac{\sigma_j}{\sigma_i}\frac{1}{r_{ij}}\Delta s_j\omega_j$$

$$C_{ij} = \left(\frac{1}{r_{ij}} + H_{ij}\right)\Delta s_j\omega_j. \tag{8.122}$$

8.8.4 Results and discussion

The results of calculation for nondimensional added mass, $\mu_{ij} = \alpha_{ij}\left(\frac{4}{3}\rho\pi r_1^3\right)^{-1}$, and nondimensional damping coefficient, $\lambda_{ij} = \alpha_{ij}\left(\frac{4}{3}\rho\omega\pi r_1^3\right)^{-1}$, for the surge and heave motions of a sphere are shown in Figures 8.6 to 8.9 as a function of Kr_1 for $\frac{h}{r_1} = 1.5$. The solid lines are the analytical solutions derived by the method of multipole expansion by Rahman [17].

The calculations by FPM were performed by subdividing the surface of the sphere into 512, 1024, and 3200 triangular panels. The figures show that it is necessary to have a large number of elements to find accurate solutions. This is a

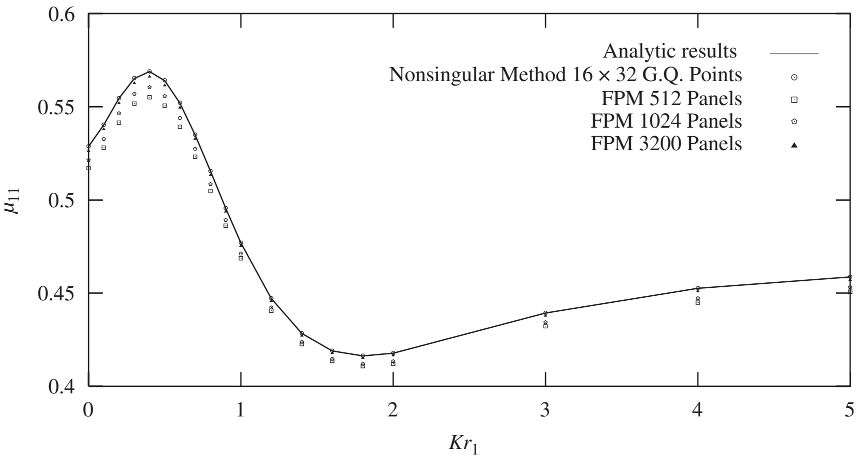

Figure 8.6: Surge added mass calculated with different methods in $h/r_1 = 1.5$.

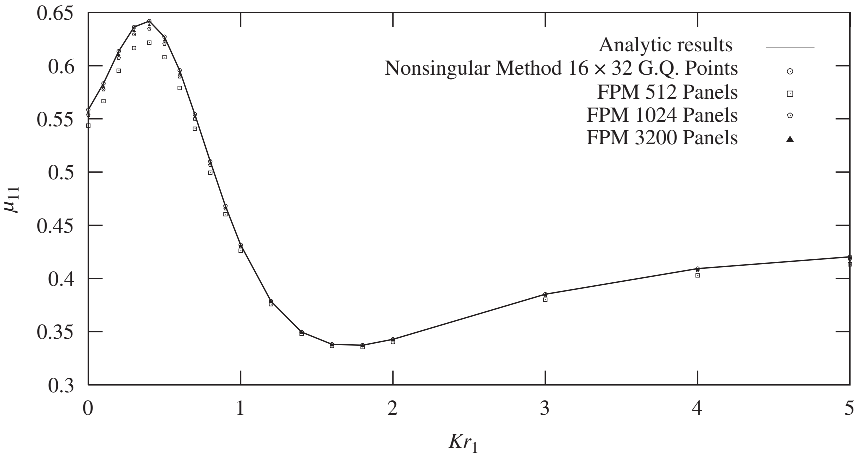

Figure 8.7: Heave added mass calculated with different methods in $h/r_1 = 1.5$.

drawback of the FPM. The solutions with the nonsingular method were computed by distributing of 16×32 Gauss–Legendre quadrature points at the surface of the sphere. The comparison of the results with this method and with the analytical results obtained by Rahman [16] show that the differences between them are at the fourth decimal point. This shows that the nonsingular method with much less discretization gives better results that the FPM. This is due to using the exact geometry in the calculation of the hydrodynamic characteristics of the flow.

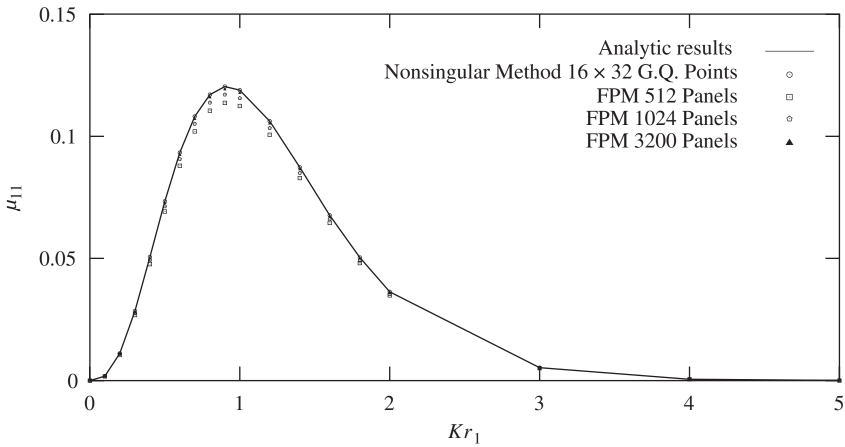

Figure 8.8: Surge damping coefficients calculated with different methods in $h/r_1 = 1.5$.

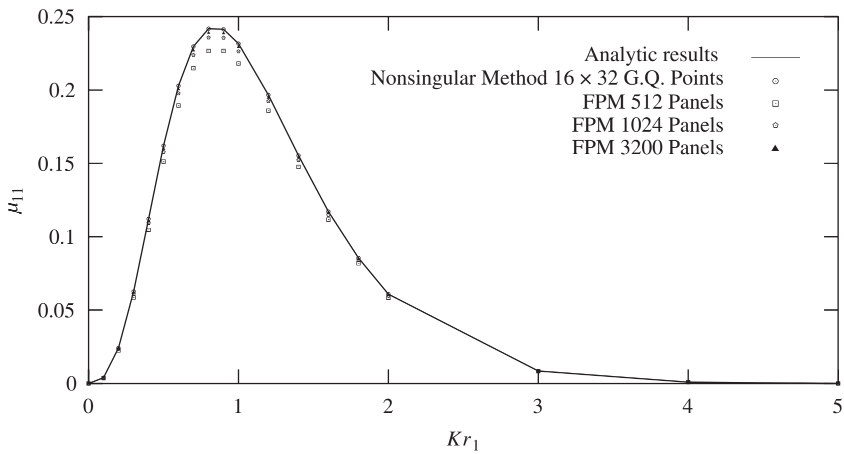

Figure 8.9: Heave damping coefficients calculated with different methods in $h/r_1 = 1.5$.

The relative error for the added mass in surge and heave motion are calculated and shown in Figures 8.10 and 8.11. The relative error is defined as

$$\text{Relative error} = \frac{|\text{numerical result} - \text{analytical result}|}{\text{analytical result}} \tag{8.123}$$

These figures also demonstrate the accuracy of the nonsingular method.

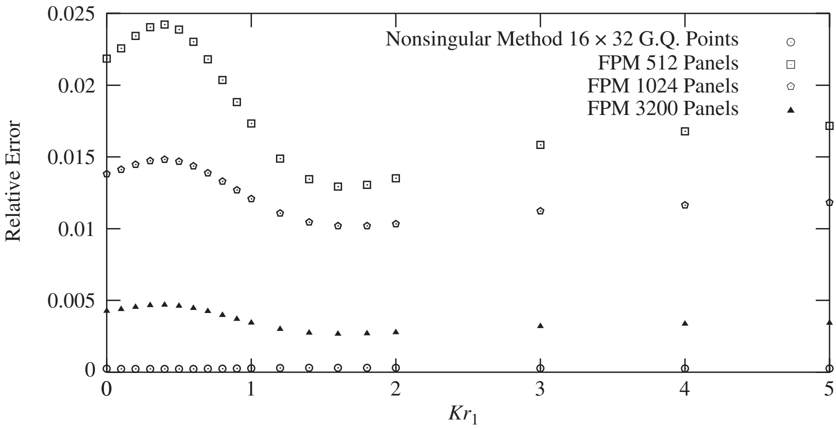

Figure 8.10: Relative error for surge added mass in $h/r_1 = 1.5$.

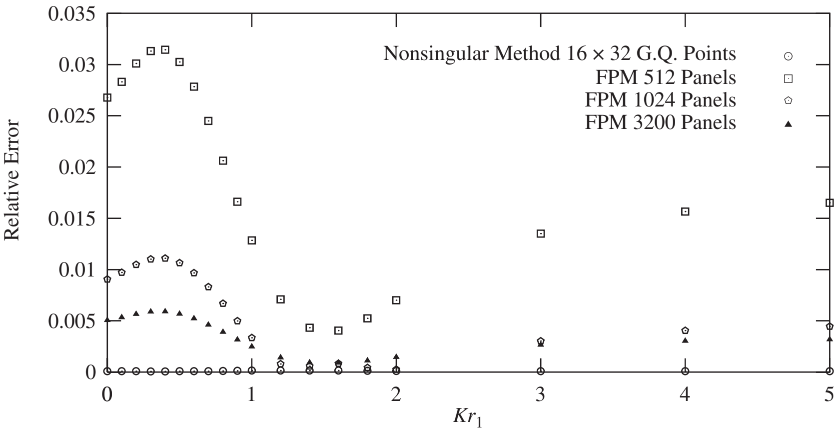

Figure 8.11: Relative error for heave added mass in $h/r_1 = 1.5$.

8.9 Seismic response of dams

In this section, we investigate an analytical solution of the seismic response of dams in one-dimension. In a real-life situation, the seismic problem is of three-dimensional nature. However, for an idealized situation, we can approximate the problem to study in one-dimension. The results obtained in this section should be treated as preliminary to obtain the prior information of the real problem.

8.9.1 Introduction

In order to analyze the safety and stability of an earth dam during an earthquake, we need to know the response of the dam to earthquake ground motion so that the

inertia forces that will be generated in the dam by the earthquake can be derived. Once the inertia forces are known, the safety and the stability of the structure can be determined. A geometrical configuration of a dam is sketched in Figure 8.12.

The inertia forces generated during an earthquake will depend on

i) The geometry of the dam
ii) The material properties
iii) The earthquake time history

The reality of the problem is that an earth dam is a three-dimensional structure, usually multizoned with variable properties in each zone. The material properties are nonlinear inelastic and the earthquake time history is a time-varying phenomenon. The problem is, therefore, very complex and a proper solution requires the use of a finite element program, which can deal with nonlinear inelastic material properties. The earthquake is the travelling-wave phenomenon, which arrives at the base of the dam through the foundation rock. Since the foundation is not rigid, part of the energy, which vibrates the dam, is lost through the foundation causing radiation damping. The analytical solution to such a problem is not at all possible. However, it is often necessary to have approximate solutions that can be used to understand the behaviour of the dam during earthquakes. In order to make the problem amenable to analytical solution, some approximations are made to create a mathematical model. Such a model in this case is known as the shear beam model (SB) of earth dams. There are several such solutions that differ in their approximations to the problem.

8.9.2 Mathematical formulation

The first set of approximations refers to the geometry of the dam.

Assumptions:

- The length of the dam is large compared to height. In this case, the presence of the abutments will not be felt except near the ends ($L > 4H$). This removes the third dimension and deals with the dam cross-section only.
- Slopes of the dam are fairly flat and the section is symmetrical about the z-axis.
- Amount of oscillations due to bending is small. The shear strain and shear stress along a horizontal line is the same everywhere and is equal to the average at the corresponding level. Therefore, subjected to horizontal loading in shear, the response is assumed to be in shear only.
- The wedge is rigidly connected to the base. The rigidity of the foundation material is much greater than that of the dam.
- The base is acted upon by an arbitrary disturbance given by the displacement function $a(t)$ in the horizontal direction only. With the above assumptions, only the z dimension and the shear stress is pertinent. Therefore, it is called **one-dimension shear-beam** analysis.

- As regards the material properties, the material in the wedge is assumed to be viscoelastic.

For homogeneous material properties with constant shear modulus, the solution is readily available, see Refs. [1], [2], [19], and [20]. For nonhomogeneous shear modulus for the material, when the modulus varies with the level, an analytical solution is given by Dakoulas and Gazetas [4]. However, the solution as given in the paper appears to have some drawbacks. The error may simply be in printing the equation in the paper or it may be a genuine error. The solution for the radiation damping is not clearly explained in their paper. Our aim in this section is to solve the equations using a different technique from that used by Dakoulas and Gazetas and check their solution.

Dakoulas and Gazetas used the nonhomogeneous model for material properties in which the shear modulus is a function of the level z. This gives:

$$G(z) = G_b(z/H)^m, \tag{8.124}$$

where

$$\begin{aligned} G_b &= \text{shear modulus at the base of the dam, and} \\ m &= \text{a parameter of the model.} \end{aligned}$$

It is generally assumed that $m = 0.5$ but experimental data suggests that m can vary considerably. It is to be noted that in a dam the depth of material at a given level varies and the above equation generally relates to a constant overburden pressure corresponding to the level z as in a soil layer. Dakoulas and Gazetas provide a detailed examination and verification of the model.

In a viscoelastic material, the shear stress τ relates to the shear strain and the strain rate by the following:

$$\tau = G(z)\left[\frac{\partial u}{\partial z} + \eta\frac{\partial \dot{u}}{\partial z}\right]. \tag{8.125}$$

In this expression, the displacement $u(z, t)$ is measured relative to the base at $z = H$. Therefore, referring to Figure 8.12, the shear force Q at the depth z is

$$Q = 2Bz\tau.$$

Hence, considering an elemental slice of width dz, the net force acting on the element is dQ and therefore the net force must equal the inertia force on the element. Thus, considering the equilibrium of the element

$$2Bzdz\rho(\ddot{u} + \ddot{a}) = \left\{2Bz\frac{\partial}{\partial z}\left[G(z)\left(\frac{\partial u}{\partial z} + \eta\frac{\partial \dot{u}}{\partial z}\right)\right] + 2B\left[G(z)\left(\frac{\partial u}{\partial z} + \eta\frac{\partial \dot{u}}{\partial z}\right)\right]\right\} dz$$

$$\text{or } \rho(\ddot{u} + \ddot{a}) = \frac{\partial}{\partial z}\left[G(z)\left(\frac{\partial u}{\partial z} + \eta\frac{\partial \dot{u}}{\partial z}\right)\right] + \frac{1}{z}\left[G(z)\left(\frac{\partial u}{\partial z} + \eta\frac{\partial \dot{u}}{\partial z}\right)\right]. \tag{8.126}$$

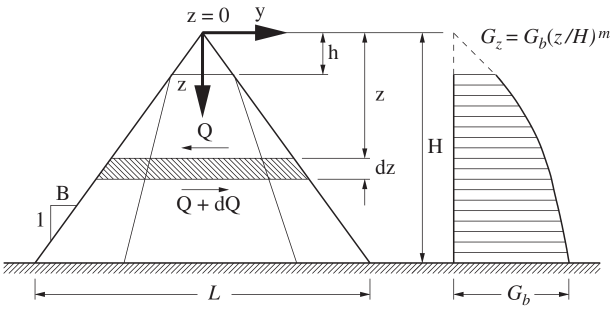

Figure 8.12: Dam cross-section and distribution of shear modulus with depth.

In the above equation, $\ddot{u}+\ddot{a}$ represents the absolute acceleration, where $\ddot{a}$ is the base acceleration. In earthquake engineering, the acceleration is the basic data and not the displacements. Using equation (8.126) and noting that $G_b/\rho = C_b^2$ where C_b is the shear wave velocity at the base level, we obtain

$$(\ddot{u}+\ddot{a}) = C_b^2\left(\frac{z}{H}\right)^m\left[\left(\frac{\partial^2 u}{\partial z^2}+\eta\frac{\partial^2\dot{u}}{\partial z^2}\right)+\left(\frac{m+1}{z}\right)\left(\frac{\partial u}{\partial z}+\eta\frac{\partial \dot{u}}{\partial z}\right)\right]. \quad (8.127)$$

The boundary conditions are $u=0$ at $z=H$, the dam is rigidly connected to the base, and $\tau=0$ at $z=h$, the top surface is stress free. This condition is derived from the shear beam model of the dam. The initial conditions are at rest, i.e. $u=\dot{u}=0$ at $t=0$ for all $h\le z\le H$.

8.9.3 Solution

This initial boundary value problem can easily be solved by using the Laplace transform method. The definition of the Laplace transform is available in any standard engineering textbook. And here we cite the definition for ready reference only (see Rahman [18])

$$\mathcal{L}\{u(z,t)\} = \int_0^\infty u(z,t)e^{-st}dt = \overline{u}(z,s),$$

and its inverse is given by

$$u(z,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\overline{u}(z,s)e^{st}ds.$$

Applying the Laplace transform to equation (8.127), we obtain

$$s^2\overline{u}+\overline{a} = C_b^2(1+\eta s)\left(\frac{z}{H}\right)^m\left[\frac{\partial^2\overline{u}}{\partial z^2}+\left(\frac{m+1}{z}\right)\frac{\partial\overline{u}}{\partial z}\right] \quad (8.128)$$

where s is the Laplace parameter and

$$\overline{u} = \mathcal{L}(u) \quad \text{and} \quad \overline{a} = \mathcal{L}(\ddot{a}), \tag{8.129}$$

in which $\mathcal{L}()$ represents Laplace operation on ().

Changing the dependent variable $\overline{u}$ to U where

$$U = z^{m/2}\overline{u} \tag{8.130}$$

$$\text{and} \quad k = \frac{sH^{m/2}}{C_b\sqrt{1+\eta s}}, \tag{8.131}$$

leads to

$$z^{m/2-2}\left[z^2\frac{\partial^2 U}{\partial z^2} + z\frac{dU}{dz} - \left(\frac{m^2}{4} + k^2 z^{2-m}\right)U\right] = \frac{\overline{a}k^2}{s^2}. \tag{8.132}$$

The complementary function is obtained from the solution of

$$z^2\frac{\partial^2 U}{\partial z^2} + z\frac{dU}{dz} - \left(\frac{m^2}{4} + k^2 z^{2-m}\right)U = 0. \tag{8.133}$$

Changing the independent variable z to μ where

$$\mu = 2k/(2-m)z^{1-m/2} \tag{8.134}$$

leads to

$$\mu^2\frac{\partial^2 U}{\partial \mu^2} + \mu\frac{dU}{d\mu} - (\mu^2 + q^2)U = 0, \tag{8.135}$$

where

$$q = m/(2-m), \tag{8.136}$$

and therefore

$$\mu = k(1+q)z^{1-m/2}. \tag{8.137}$$

The solution is, therefore,

$$U = AI_q(\mu) + BK_q(\mu). \tag{8.138}$$

The particular integral is

$$\overline{u} = -\frac{\overline{a}}{s^2}, \tag{8.139}$$

and therefore the complete solution is

$$\overline{u} = z^{-m/2}[AI_q(\mu) + BK_q(\mu)] - \frac{\overline{a}}{s^2} \tag{8.140}$$

$$\tau = C_b^2(1+\eta s)\rho\frac{\partial \overline{u}}{\partial z}$$
$$= C_b^2(1+\eta s)\rho\{[z^{-m}(AI_q'(\mu)+BK_q'(\mu))]k(1+q)(1-m/2)$$
$$-\frac{m}{2}z^{-1-m/2}(AI_q(\mu)+BK_q(\mu))\}. \tag{8.141}$$

Note that:

$$I_q'(\mu) = I_{q+1}(\mu) + \tfrac{q}{\mu}{}_q(\mu) \text{ and } K_q'(\mu) = -K_{q+1}(\mu) + \tfrac{q}{\mu}K_q(\mu). \tag{8.142}$$

Applying the two boundary conditions, i.e. at $z = h, \tau = 0$ gives

$$AI_{q+1}(\mu_h) - BK_{q+1}(\mu_h) = 0 \tag{8.143}$$

and at $z = H, u = 0$ gives

$$H^{-m/2}(AI_q(\mu_H) + BK_q(\mu_H)) = \frac{\overline{a}}{s^2}. \tag{8.144}$$

Therefore, solving for A and B from equations (8.143) and (8.144) and replacing in equation (8.140), the solution becomes

$$\overline{u} = \frac{\overline{a}}{s^2}\left[\left(\frac{z}{H}\right)^{-m/2}\frac{[K_{q+1}(\mu_h)I_q(\mu)+I_{q+1}(\mu_h)K_q(\mu)]}{[K_{q+1}(\mu_h)I_q(\mu_H)+I_{q+1}(\mu_h)K_q(\mu_H)]} - 1\right], \tag{8.145}$$

where $\mu_h = k(1+q)h^{1-m/2}$ and $\mu_H = k(1+q)H^{1-m/2}$. The solution u will be obtained from the Laplace inversion of equation (8.145). Equation (8.145) can be written as

$$\overline{u} = \overline{a}F(z,s). \tag{8.146}$$

From equation (8.129),

$$\mathcal{L}^{-1}(\overline{a}) = \ddot{a}(t).$$

If

$$\mathcal{L}^{-1}(F) = f(z,t), \tag{8.147}$$

then using the convolution integral,

$$u = \int_0^t \ddot{a}(\tau)f\{(t-\tau),z\}d\tau. \tag{8.148}$$

The problem is to solve equation (8.147) to obtain $f(z,t)$.

$$F = \frac{1}{s^2}\left[\left(\frac{z}{H}\right)^{-m/2} \frac{[K_{q+1}(\mu_h)I_q(\mu) + I_{q+1}(\mu_h)K_q(\mu)]}{[K_{q+1}(\mu_h)I_q(\mu_H) + I_{q+1}(\mu_h)K_q(\mu_H)]} - 1\right]. \tag{8.149}$$

The inverse of the transform can be obtained by using the residue calculus

$$\begin{aligned} f(z,t) &= \mathcal{L}^{-1}\{F(z,s)\} \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(z,s)e^{st}\,ds \\ &= \sum \text{Residues at the poles of } F(z,s)e^{st}. \end{aligned}$$

It can be easily noted that $s=0$ is a double pole. The other poles can be determined from the roots of the following equation

$$K_{q+1}(\mu_h)I_q(\mu_H) + I_{q+1}(\mu_h)K_q(\mu_H) = 0. \tag{8.150}$$

First, we determined the residue at the double pole $s=0$, which is found to be $R_0 = t$. Next the residues corresponding to the single poles

$$\begin{aligned} R_n &= \frac{Q(\omega_n)}{\omega_n P(\omega_n)}(ie^{i\omega_n t}) \\ \text{and} \qquad R_n^* &= \frac{Q(\omega_n)}{\omega_n P(\omega_n)}(ie^{-i\omega_n t}). \end{aligned} \tag{8.151}$$

Therefore, the total residues are

$$\begin{aligned} R_n + R_n^* &= \frac{Q(\omega_n)}{\omega_n P(\omega_n)}[i(e^{i\omega_n t} - e^{-i\omega_n t}] \\ &= -2\frac{Q(\omega_n)}{P(\omega_n)}\left(\frac{\sin \omega_n t}{\omega_n t}\right). \end{aligned} \tag{8.152}$$

Detailed calculations of the residues can be found in the work of Rahman [22].

The solution for $f(z,t)$ can be written as

$$f(z,t) = -2\{z/H\}^{-m/2}\sum_{n=1}^{\infty}\frac{Q(\omega_n)\sin(\omega_n t)}{P(\omega_n)\omega_n}, \tag{8.153}$$

where

$$\begin{aligned} Q(\omega_n) &= J_q(a\xi^{1-m/2})Y_{q+1}(a\lambda^{1-m/2}) - Y_q(a\xi^{1-m/2})J_{q+1}(a\lambda^{1-m/2}) \\ P(\omega_n) &= a\lambda^{1-m/2}\{J_q(a\lambda^{1-m/2})Y_q(a) - Y_q(a\lambda^{1-m/2})J_q(a)\} \\ &\quad -a\{J_{q+1}(a)Y_{q+1}(a\lambda^{1-m/2}) - Y_{q+1}(a)J_{q+1}(a\lambda^{1-m/2})\}, \end{aligned} \tag{8.154}$$

and the characteristic equation is

$$J_{q+1}(a\lambda^{1-m/2})Y_q(a) - J_q(a)Y_{q+1}(a\lambda^{1-m/2}) = 0, \tag{8.155}$$

in which $\xi = z/H$, $a = \frac{\omega_n H}{C_b}(1+q)$, and $\lambda = h/H$. Once we know the time history of the earthquake, we can obtain the complete solution by the convolution integral equation (8.148).

8.10 Transverse oscillations of a bar

The transverse oscillations in a bar is a technically important problem. This problem can be studied under very general conditions by considering the influence function $G(x,t)$. This influence function is usually called the Green's function. Mathematically, we define it as a kernel of the problem.

Let us suppose that, in its state of rest, the axis of the bar coincides with the segment $(0,\ell)$ of the x-axis and that the deflection parallel to the z-axis of a point of x at time t is $z(x,t)$ (see Figure 8.13). This is governed by the following integro-differential equation

$$z(x,t) = \int_0^\ell G(x,\eta)\left[p(\eta) - \mu(\eta)\frac{\partial^2 z}{\partial t^2}\right]d\eta, \quad (0 \le x \le \ell), \tag{8.156}$$

where $p(\eta)d\eta$ is the load acting on the portion $(\eta, \eta + d\eta)$ of the bar in the direction of Oz, and $\mu(\eta)d\eta$ the mass of the portion.

In particular, in the important case of harmonic vibrations

$$z(x,t) = Z(x)e^{i\omega t} \tag{8.157}$$

of an unloaded bar ($p(\eta) = 0$), we obtain

$$Z(x) = \omega^2 \int_0^\ell G(x,\eta)\mu(\eta)Z(\eta)d\eta, \tag{8.158}$$

which is the homogeneous integral equation. Equation (8.158) shows that our vibration problem belongs properly to the theory of Fredholm integral equations. In some cases it is possible to obtain quite precise results, even by means of the more elementary theory of Volterra integral equations. For instance, this happens in the

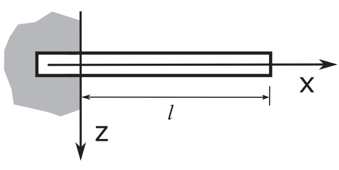

Figure 8.13: Transverse oscillations of a bar.

case of uniform bar $\mu(x) = \mu =$ constant clamped at the end $x = 0$ and free at the end $x = \ell$. Its transverse oscillations are governed by the partial differential equation

$$\frac{\partial^4 z}{\partial x^4} + \frac{\mu}{j}\frac{\partial^2 z}{\partial t^2} = 0 \tag{8.159}$$

where $j = EI$ is the constant bending rigidity of the bar, together with

$$\begin{aligned} z(0,t) &= \frac{\partial z}{\partial x}(0,t) = 0 \\ \frac{\partial^2 z}{\partial x^2}(\ell,t) &= \frac{\partial^3 z}{\partial x^3}(\ell,t) = 0 \end{aligned} \tag{8.160}$$

as conditions on the ends. If use is made of the previous statement, i.e. equation (8.157), we obtain for the transverse harmonic vibrations of frequency

$$\nu = \frac{\omega}{2\pi} \tag{8.161}$$

the ordinary differential equation

$$\frac{d^4 Z}{dx^4} - k^4 Z = 0 \quad \text{with} \quad k^4 = 4\pi^2\nu^2\frac{\mu}{j} \tag{8.162}$$

together with the end conditions

$$\begin{aligned} Z(0) &= Z'(0) = 0 \\ Z''(\ell) &= Z'''(\ell) = 0. \end{aligned} \tag{8.163}$$

Neglecting momentarily the second condition of equation (8.163) at $x = \ell$, equation (8.162) together with the condition of equation (8.163) at $x = 0$, can be transformed into a Volterra integral equation of the second kind by integrating successively. In this way, if we put $Z''(0) = c_2, Z'''(0) = c_3$ we obtain the equation

$$\phi(x) = k^4 \int_0^x \frac{(x-t)^3}{3!}\phi(t)dt + k^4\left(c_2\frac{x^2}{2!} + c_3\frac{x^3}{3!}\right) \tag{8.164}$$

Hence we obtain an equation with special kernel. Now to evaluate the function $\phi(x)$, we use the Laplace transform method. Thus, taking the Laplace transform of both side of equation (8.164) and after a little reduction yields

$$\begin{aligned} \mathcal{L}\{\phi(x)\} &= k^4\left\{c_2\left(\frac{s}{s^4-k^4}\right) + c_3\left(\frac{1}{s^4-k^4}\right)\right\} \\ &= k^4\left\{\frac{c_2 s}{2k^2}\left[\frac{1}{s^2-k^2} - \frac{1}{s^2+k^2}\right] + \frac{c_3}{2k^2}\left[\frac{1}{s^2-k^2} - \frac{1}{s^2+k^2}\right]\right\} \end{aligned}$$

The Laplace inverse of the above transform yields

$$\begin{aligned}\phi(x) &= \frac{c_2k^2}{2}(\cosh kx - \cos kx) + \frac{c_3k}{2}(\sinh kx - \sin kx)\\ &= \alpha(\cosh kx - \cos kx) + \beta(\sinh kx - \sin kx) \qquad (8.165)\end{aligned}$$

where α and β are the two redefined constants. It is interesting to note that solving the fourth-order differential equation (8.162) we obtain the same solution, i.e. equation (8.165). Now satisfying the other two conditions at $x = \ell$, we obtain two equations to determine the constants α and β. And they are

$$\begin{aligned}\alpha(\cosh k\ell + \cos k\ell) + \beta(\sinh k\ell + \sin k\ell) &= 0\\ \alpha(\sinh k\ell - \sin k\ell) + \beta(\cosh k\ell + \cos k\ell) &= 0\end{aligned}$$

Thus, if the determinant of the coefficients is different from zero, the unique solution of this system is $\alpha = \beta = 0$, and the corresponding solution of the given equation is the trivial one

$$\phi(x) = Z(x) = 0;$$

but if the determinant vanishes, then there are also nontrivial solutions to our problem. This shows that the only possible harmonic vibrations of our bar are those which correspond to the positive roots of the transcendental equation

$$\sinh^2(k\ell) - \sin^2(k\ell) = [\cosh(k\ell) + \cos(k\ell)]^2,$$

that is, after a little reduction,

$$\cosh\xi\cos\xi + 1 = 0 \qquad (8.166)$$

where

$$\xi = k\ell \qquad (8.167)$$

This transcendental equation can be solved graphically (see Figure 8.14) by means of the intersection of the two curves

$$\eta = \cos\xi \text{ and } \eta = -\frac{1}{\cosh\xi}.$$

From the graphical solution, we find explicitly the successive positive roots $\xi_1, \xi_2 \ldots$ of equation (8.166) as

$$\xi_1 = 1.875106, \ \ \xi_2 = 4.6941, \ldots.$$

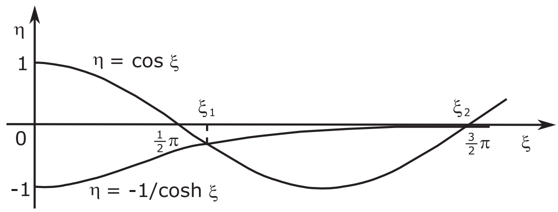

Figure 8.14: Graphical solution of equation (8.166).

By means of equation (8.167) and the second equation of equation (8.162), we obtain the corresponding natural frequencies of our bar

$$\nu_n = \frac{1}{2\pi}\sqrt{\left(\frac{j}{\mu}\right)\frac{\xi_n^2}{\ell^2}} \quad (n = 1, 2, 3, \ldots) \tag{8.168}$$

Contrary to what happens in the case of a vibrating string, these frequencies are not successive multiples of the first and are inversely proportional to the square of ℓ, instead of ℓ itself. In spite of the thoroughness of the previous results, its interest from the point of view of the theory of integral equations is not great because the key formula, i.e. equation (8.165) for $\phi(x)$ (or the equivalent one for $Z(x)$) can also be obtained directly (and more rapidly) from the linear equation with constant coefficients, i.e. equation (8.162).

From the point of view of the theory of integral equations, it is more interesting that from the integral equation (8.164) itself we can deduce a good approximation for the first eigenvalue

$$k_1^4 = \left(\frac{\xi_1}{\ell}\right)^4 = \frac{12.362}{\ell^4},$$

because the same device can be used even if the integral equation is not explicitly solvable.

8.11 Flow of heat in a metal bar

In this section, we shall consider a classical problem of heat flow in a bar to demonstrate the application of integral equations. The unsteady flow of heat in one-dimensional medium is well-known. The governing partial differential equation with its boundary and initial conditions are given below.

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} - qT, \tag{8.169}$$

the boundary conditions are

$$T(0,t) = T(\ell,t)$$
$$\frac{\partial T}{\partial x}(0,t) = \frac{\partial T}{\partial x}(\ell,t) \tag{8.170}$$

and the initial condition is

$$T(x,0) = f(x). \tag{8.171}$$

where $T(x,t)$ is the temperature at any time and at any position of the bar, α the thermal diffusivity, q the heat loss through the bar, $x=0$ the left-hand side of the bar and $x=\ell$ the right-hand side of the bar, and ℓ the length of the bar. t is the time.

This problem can be solved by using the separation of variable method. There are two ways to select the separation constant, and this constant will later on turn out to be the eigenvalue of the problem with the eigenfunction as the solution. Our aim is to find a kernel of the ordinary differential equation which will represent the Green's function or simply the influence function of the problem.

Solution of the problem

Let us consider the product solution in the following manner (see Rahman [15, 16])

$$T(x,t) = u(x)\phi(t) \tag{8.172}$$

Using this product solution into the equations, we obtain from equation (8.169)

$$\frac{\phi'(t)}{\alpha\phi(t)} = \frac{u''(x)}{u(x)} - h^2 = -\lambda \tag{8.173}$$

where $\frac{q}{\alpha} = h^2$, and λ is a separation constant. The two ordinary differential equations can be written as

$$\frac{d^2u}{dx^2} + (\lambda - h^2)u = 0 \tag{8.174}$$
$$\frac{d\phi}{dt} + \alpha\lambda\phi = 0 \tag{8.175}$$

The separation constant ($\lambda > 0$) is assumed to be positive to obtain an exponentially decaying time solution for $\phi(t)$. The boundary conditions are

$$u(0) = u(\ell); \quad u'(0) = u'(\ell) \tag{8.176}$$

There are three cases to investigate:

Case I

Let us consider $\lambda - h^2 = \nu^2 = 0$, that means $\lambda = h^2$. Here, ν is a real number. Thus, ν^2 is a positive real number.

The general solutions of equations (8.174) and (8.175) are given by

$$\begin{aligned} u(x) &= A + Bx \\ \phi(t) &= Ce^{-\alpha h^2 t} \end{aligned} \tag{8.177}$$

The boundary conditions, i.e. equation (8.176) yields that

$$u(x) = A \tag{8.178}$$

Thus, the temperature solution is

$$T_0(x) = ACe^{-\alpha h^2 t} = A_0 e^{-\alpha h^2 t}. \tag{8.179}$$

where A_0 is a redefined constant.

Case II

Let us consider $\lambda - h^2 = \nu^2 > 0$. The general solutions of equation (8.174) and (8.175) are given by

$$\begin{aligned} u(x) &= A \cos \nu x + B \sin \nu x \\ \phi(t) &= Ce^{-\alpha \lambda t} \end{aligned} \tag{8.180}$$

The boundary conditions, i.e. equation (8.176) give the following two equations for the determination of A and B.

$$\begin{aligned} A(1 - \cos \nu\ell) - B \sin \nu\ell &= 0 \\ A \sin \nu\ell + B(1 - \cos \nu\ell) &= 0 \end{aligned} \tag{8.181}$$

These two equations are compatible for values of A, B both not zero, if and only if

$$\begin{vmatrix} 1 - \cos \nu\ell & -\sin \nu\ell \\ \sin \nu\ell & 1 - \cos \nu\ell \end{vmatrix} = 0.$$

That is, if $2(1 - \cos \nu\ell) = 0$, which implies that

$$\nu = \frac{2n\pi}{\ell}.$$

For these values of ν, equation (8.180) is satisfied by all values of A and B. For other values of ν, the only solution is the trivial one $u(x)=0$. The eigenvalues are then

$$\lambda_n = h^2 + \frac{4n^2\pi^2}{\ell^2}, \quad n = 1, 2, 3, \ldots.$$

Hence eigenfunction solutions are

$$u_n(x) = A_n \cos\frac{2n\pi x}{\ell} + B_n \sin\frac{2n\pi x}{\ell}.$$

The $\phi(t)$ solution is

$$\phi(t) = Ce^{-\alpha(h^2+\frac{4n^2\pi^2}{\ell^2})t}.$$

Hence the temperature solution is

$$\begin{aligned} T_n(x,t) &= \left[A_n \cos\frac{2n\pi x}{\ell} + B_n \sin\frac{2n\pi x}{\ell}\right]\left[Ce^{-\alpha(h^2+\frac{4n^2\pi^2}{\ell^2})t}\right], \quad (n = 1, 2, \ldots) \\ &= \left[A_n \cos\frac{2n\pi x}{\ell} + B_n \sin\frac{2n\pi x}{\ell}\right]\left[e^{-\alpha(h^2+\frac{4n^2\pi^2}{\ell^2})t}\right], \quad (n = 1, 2, \ldots). \end{aligned} \tag{8.182}$$

Here, the constant C is merged with A_n and B_n. The complete solution up to this point can be written as

$$\begin{aligned} T(x,t) &= \sum_{n=0}^{\infty} T_n(x,t) \\ &= \sum_{n=0}^{\infty}\left[A_n \cos\frac{2n\pi x}{\ell} + B_n \sin\frac{2n\pi x}{\ell}\right]\left[e^{-\alpha(h^2+\frac{4n^2\pi^2}{\ell^2})t}\right], \quad (n = 1, 2, \ldots). \end{aligned} \tag{8.183}$$

Case III

Let us consider $\lambda - h^2 = -\nu^2 < 0$. The general solutions of equations (8.174) and (8.175) are given by

$$\begin{aligned} u(x) &= A\cosh \nu x + B\sinh \nu x \\ \phi(t) &= Ce^{-\alpha\lambda t} \end{aligned} \tag{8.184}$$

The boundary conditions, i.e. equation (8.176) give the following two equations for the determination of A and B.

$$\begin{aligned} A(1 - \cosh \nu\ell) + B\sinh \nu\ell &= 0 \\ -A\sinh \nu\ell + B(1 - \cosh \nu\ell) &= 0 \end{aligned} \tag{8.185}$$

These two equations are compatible for values of A, B both not zero, if and only if

$$\begin{vmatrix} 1-\cosh \nu\ell & \sinh \nu\ell \\ -\sinh \nu\ell & 1-\cosh \nu\ell \end{vmatrix} = 0.$$

That is, if $\cosh \nu\ell(1-\cosh \nu\ell)=0$, which implies that either $\cosh \nu\ell = 0$ or $\cosh \nu\ell = 1$. For real ν the first condition cannot be satisfied, and the second condition can be satisfied only when $\nu = 0$. However, $\nu = 0$ is considered already in case (I). Thus, only possibility is that both A and B must vanish implying that the solution is a trivial one $u(x)=0$. The eigenvalues of the problem are then $\lambda_n = h^2 + \frac{4n^2\pi^2}{\ell^2}, n = 0, 1, 2, \ldots$ with normalized eigenfunctions

$$\frac{1}{\sqrt{\ell}}, \sqrt{\frac{2}{\ell}} \cos \frac{2n\pi x}{\ell}, \sqrt{\frac{2}{\ell}} \sin \frac{2n\pi x}{\ell}, \cdots,$$

and since $\int_0^\ell \cos \frac{2n\pi x}{\ell} \sin \frac{2n\pi x}{\ell} dx = 0$, the complete normalized orthogonal system is

$$\frac{1}{\sqrt{\ell}}, \sqrt{\frac{2}{\ell}} \cos \frac{2n\pi x}{\ell}, \sqrt{\frac{2}{\ell}} \sin \frac{2n\pi x}{\ell}, \cdots,$$

with the eigenvalues $\lambda_n = h^2 + \frac{4n^2\pi^2}{\ell^2}, n = 0, 1, 2, \ldots$.

Thus, the complete solution of the problem is to satisfy the initial condition $T(x, 0) = f(x)$. Therefore, using this condition, we obtain

$$f(x) = \sum_{n=0}^{\infty} \left(A_n \cos \frac{2n\pi x}{\ell} + B_n \sin \frac{2n\pi x}{\ell} \right) \tag{8.186}$$

which is a Fourier series with Fourier coefficients A_n and B_n. These coefficients are determined as follows:

$$\begin{aligned} A_n &= \frac{2}{\ell} \int_0^\ell f(x) \cos \frac{2n\pi x}{\ell}\, dx \\ B_n &= \frac{2}{\ell} \int_0^\ell f(x) \sin \frac{2n\pi x}{\ell}\, dx \\ A_0 &= \frac{1}{\ell} \int_0^\ell f(x) dx. \end{aligned}$$

Equivalence with the integral equation

The ordinary differential equation is

$$u''(x) + (\lambda - h^2)u = 0$$

and so the Green's function must satisfy the equation

$$K_{xx} - h^2 K = -\delta(x - \eta).$$

The boundary conditions for the Green's function are

$$\begin{aligned} K(0) &= K(\ell); \\ K_x(0) &= K_x(\ell); \\ K(\eta - 0) &= K(\eta + 0); \\ K_x(\eta - 0) - K_x(\eta + 0) &= 1. \end{aligned}$$

The solution is written as

$$K(x, \eta) = \begin{cases} A \cosh(hx) + B \sinh(hx), & 0 \leq x \leq \eta \\ C \cosh(hx) + D \sinh(hx), & \eta \leq x \leq \ell \end{cases} \tag{8.187}$$

Applying the first two boundary conditions we obtain

$$K(x, \eta) = \begin{cases} C \cosh(hx + \ell h) + D \sinh(hx + \ell h), & 0 \leq x \leq \eta \\ C \cosh(hx) + D \sinh(hx), & \eta \leq x \leq \ell \end{cases} \tag{8.188}$$

Using the continuity condition at $x = \eta$ (the third boundary condition), the constants C and D can be determined as

$$\begin{aligned} C &= \gamma\,(\sinh(\eta h) - \sinh(\eta h + \ell h)) \\ D &= \gamma\,(\cosh(\eta h + \ell h) - \cosh(\eta h)) \end{aligned}$$

Substituting the values of C and D in equation (8.188) and after simplifications we obtain the Green's function as

$$K(x, \eta) = \begin{cases} \gamma\,(\sinh(\eta h - hx - \ell h) - \sinh(\eta h - hx)), & 0 \leq x \leq \eta \\ \gamma\,(\sinh(xh - \eta h - \ell h) - \sinh(xh - \eta h)), & \eta \leq x \leq \ell \end{cases}$$

Next using the jump discontinuity condition (the fourth boundary condition) we find that the value of γ is simply

$$\gamma = \frac{1}{2h(1 - \cosh(\ell h))}.$$

Hence inserting the value of γ in the above equation and after some reduction, we can construct the Green's function as follows:

$$K(x, \eta) = \begin{cases} \frac{\cosh h(\eta - x - \frac{\ell}{2})}{2h \sinh \frac{h\ell}{2}}, & 0 \leq x \leq \eta \\ \frac{\cosh h(x - \eta - \frac{\ell}{2})}{2h \sinh \frac{h\ell}{2}}, & \eta \leq x \leq \ell \end{cases} \tag{8.189}$$

and the boundary value problem is equivalent to the integral equation

$$u(x) = \lambda \int_0^{\ell} K(x,\eta)u(\eta)d\eta. \tag{8.190}$$

It is worth mentioning here that this integral equation (8.190) will be very hard to solve. It is advisable, therefore, that the boundary value problem in differential equation form will be much easier to obtain the solution of this complicated heat transfer problem.

8.12 Exercises

1. Find the function $u(x)$ such that, for $x \geq 0$

$$u(x) = \sin x - \int_0^x u(x-t)t\cos t dt.$$

2. Solve the integro-differential equation

$$\frac{du}{dx} = 3\int_0^x u(x-t)\cosh t dt$$

for $x \geq 0$, given that $u(0) = 1$.

3. Find the two solutions of the equation

$$\int_0^x f(x-t)f(t)dt = x^3$$

and show that they hold for $x < 0$ as well as $x \geq 0$.

4. Let $f(x)$ have a continuous differentiable coefficients when $-\pi \leq x \leq \pi$. Then the equation

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} u(x\sin\theta)d\theta$$

has one solution with a continuous differential coefficient when $-\pi \leq x \leq \pi$, namely

$$u(x) = f(0) + \int_0^{\pi/2} f'(x\sin\theta)d\theta.$$

5. Show that, if $f(x)$ is continuous, the solution of

$$\left(1 - \frac{1}{4}\lambda^2\pi\right)u(x) = f(x) + \lambda\int_0^{\infty} f(t)\cos(2xt)dt,$$

assuming the legitimacy of a certain change of order of integration.

6. Show that even periodic solutions (with period 2π) of the differential equation

$$\frac{d^2u}{dx^2} + (\lambda^2 + k^2\cos^2 x)u(x) = 0$$

satisfy the integral equation

$$u(x) = \lambda \int_{-\pi}^{\pi} e^{(k\cos x\cos t)} u(t)dt.$$

7. Show that the Weber–Hermite functions

$$W_n(x) = (-1)^n e^{\frac{1}{4}x^2} \frac{d^n}{dx^n}\left(e^{-\frac{1}{2}x^2}\right)$$

satisfy

$$u(x) = \lambda \int_{-\infty}^{\infty} e^{\frac{1}{2}ixt} u(t)dt$$

for the characteristic value of λ.

8. Show that, if $|h| < 1$, the characteristic functions of the equation

$$u(x) = \frac{\lambda}{2\pi}\int_{-\pi}^{\pi} \frac{1-h^2}{1-2h\cos(t-x)+h^2} u(t)dt$$

are $1, \cos m\pi, \sin m\pi$, the corresponding numbers being $1, \frac{1}{h^m}, \frac{1}{h^m}$, where m takes all positive integral values.

9. Show that the characteristic functions of

$$u(x) = \lambda \int_{-\pi}^{\pi} \left\{ \frac{1}{4\pi}(x-t)^2 - \frac{1}{2}|x-t| \right\} u(t)dt$$

are

$$u(x) = \cos mx, \sin mx,$$

where $\lambda = m^2$ and m is any integer.

10. Show that

$$u(x) = \int_0^x t^{x-t} u(t)dt$$

has the discontinuous solution $u(x) = kx^{x-1}$.

11. Show that a solution of the integral equation with a symmetric kernel

$$f(x) = \int_a^b K(x,t)u(t)dt$$

is

$$u(x) = \sum_{n=1}^{\infty} a_n \lambda^n u_n(x)$$

provided that this series converges uniformly, where λ_n, $u_n(x)$ are the eigenvalues and eigenfunctions of $K(x, t)$ and $\sum_{n=1}^{\infty} a_n u_n(x)$ is the expansion of $f(x)$.

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Appendix A Miscellaneous results

Some Fourier series

Let $f(x) = \ln 2(1 - \cos x)$ a periodic function defined in $0 < x < 2\pi$ such that $f(x+2\pi) = f(x)$. Then $f(x)$ can be expanded in terms of cos and sin functions as a Fourier series given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos(nx) + b_n \sin(nx)\}, \quad (0 < x < 2\pi).$$

The Fourier coefficients a_n and b_n can be obtained as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx \end{aligned}$$

It can be easily verified that $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx = 0$ since $f(2\pi - x) = f(x)$ in $(0 < x < 2\pi)$. The coefficient a_n can be calculated as follows:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx \\ &= \frac{1}{\pi}\int_0^{2\pi} \ln 2(1 - \cos x)\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^{2\pi} \ln\left(2\sin\frac{x}{2}\right)\cos(nx)dx \\ &= \frac{4}{\pi}\int_0^{\pi} \ln(2\sin x)\cos(2nx)dx \\ &= -\frac{2}{n\pi}\int_0^{\pi} \frac{\cos x \sin(2nx)}{\sin x}dx \end{aligned} \tag{A.1}$$

The last result is obtained using integration by parts.

Now let

$$C_n = \int_0^{\pi} \frac{\sin(2n-1)x}{\sin x} dx,$$

then it can be easily noted that

$$C_{n+1} - C_n = 2\int_0^{\pi} \cos(2nx)dx = 0.$$

Since $C_1 = \pi$, we therefore have $C_n = \pi$ and from equation (A.1)

$$a_n = -\frac{(C_{n+1} + C_n)}{n\pi} = -\frac{2}{n}.$$

We also require

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \ln 2(1-\cos x)dx = \frac{4}{\pi}\int_0^{\pi} \ln(2\sin x)dx. \tag{A.2}$$

Therefore,

$$\begin{aligned} a_0 &= \frac{8}{\pi}\int_0^{\pi/2} \ln(2\sin x)dx \\ &= \frac{8}{\pi}\int_0^{\pi/2} \ln(2\cos x)dx \end{aligned}$$

Adding these two expressions we have

$$\begin{aligned} 2a_0 &= \frac{8}{\pi}\int_0^{\pi/2} \ln(2\sin 2x)dx \\ &= \frac{4}{\pi}\int_0^{\pi} \ln(2\sin x)dx \end{aligned}$$

and reference to equation (A.2) now show that $a_0 = 0$.

We have proved that

$$-\frac{1}{2}\ln 2(1-\cos x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n}, \quad (0 < x < 2\pi).$$

It follows that

$$\ln\left\{2\sin\frac{1}{2}(x+t)\right\} = -\sum_{n=1}^{\infty}\frac{\cos\{n(x+t)\}}{n}$$
$$(0 \le x, t \le \pi, x+t \ne 0, x+t \ne 2\pi) \tag{A.3}$$

and that

$$\ln\left\{2\sin\frac{1}{2}(x-t)\right\} = -\sum_{n=1}^{\infty}\frac{\cos\{n(x-t)\}}{n}, \quad (0 \le t \le x \le \pi) \tag{A.4}$$

Adding equations (A.3) and (A.4), we find that

$$\ln\{2(\cos t - \cos x)\} = -\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)\cos(nx)\cos(nt), \quad (0 \le t < x \le \pi)$$

and, by symmetry,

$$\ln\{2|\cos x - \cos t|\} = -\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)\cos(nx)\cos(nt), \quad (0 \le x < t \le \pi).$$

Subtraction of equation (A.4) from equation (A.3) similarly gives

$$\ln\left|\frac{\sin\frac{1}{2}(x+t)}{\sin\frac{1}{2}(x-t)}\right| = \sum_{n=1}^{\infty}\left(\frac{2}{n}\right)\sin(nx)\sin(nt), \quad (0 \le t < x \le \pi).$$

It is worth noting that $\ln\{2|\cos x + \cos t|\} = -\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)(-1)^n\cos(nx)\cos(nt)$, $(0 \le x, t \le \pi, x+t \ne \pi)$ and hence that

$$\ln\left|\frac{\cos x + \cos t}{\cos x - \cos t}\right| = \sum_{n=1}^{\infty}\frac{4}{2n-1}\cos\{(2n-1)x\}\cos\{(2n-1)t\}$$
$$(0 \le x, t \le \pi, x \ne t, x+t \ne \pi).$$

A further Fourier series is easily established. First note that

$$\ln(1 - ae^{ix}) = -\sum_{n=1}^{\infty}\frac{1}{n}a^n e^{inx}, \quad (0 < a < 1, 0 \le x \le 2\pi)$$

and take real parts to give

$$\ln(1 - 2a\cos x + a^2) = -\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)a^n\cos(nx) \quad (0 < a < 1, 0 \le x \le 2\pi).$$

The gamma and beta functions

The gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}dt \quad (x > 0)$$

and the beta function is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (x > 0, y > 0).$$

It is easily shown that $\Gamma(1) = 1$ and an integration by parts gives the recurrence relation

$$\Gamma(x+1) = x\Gamma(x) \quad (x > 0),$$

from which it follows that

$$\Gamma(n+1) = n!.$$

The gamma and beta functions are connected by the relationship

$$B(x,y) = B(y,x) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (x > 0, y > 0). \tag{A.5}$$

This formula, and others following, are derived in many texts on analysis or calculus. The reflection formula for gamma functions is

$$\Gamma(x)\Gamma(1-x) = \pi \text{cosec}(\pi x) \quad (0 < x < 1) \tag{A.6}$$

and from equation (A.5) we see that

$$B(x, 1-x) = \pi \text{cosec}(\pi x) \quad (0 < x < 1).$$

The relationship

$$\Gamma(2x) = \frac{1}{\sqrt{\pi}} 2^{2x-1}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) \quad (x > 0)$$

is called the duplication formula. From this and also from equation (A.5), we see that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Other values of Γ are $\Gamma\left(\frac{1}{4}\right) = 3.6256\ldots$ and $\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi / \Gamma\left(\frac{1}{4}\right) = 1.2254\ldots$.

A further function which is sometimes useful in evaluating integrals is ψ (digamma) function:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad (x > 0) \tag{A.7}$$

The other definition is

$$\psi(x) = \int_0^\infty \left\{\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right\} dt \quad (x > 0). \tag{A.8}$$

A Cauchy principal value integral

Let

$$C_\gamma = \int_0^\infty \frac{u^{\gamma-1}}{1-u}\,du \quad (0 < \gamma < 1)$$

where the integral is to be interpreted as a Cauchy principal value. Now setting $u = e^s$ we have

$$\begin{aligned}
C_\gamma &= \int_{-\infty}^{\infty} \frac{e^{\gamma s}}{1-e^s} ds \\
&= \lim_{\varepsilon\to 0} \left\{ \int_{-\infty}^{-\varepsilon} \frac{e^{\gamma s}}{1-e^s}\, ds + \int_{\varepsilon}^{\infty} \frac{e^{\gamma s}}{1-e^s}\, ds \right\} \\
&= \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\infty} \left\{ \frac{e^{-\gamma s}}{1-e^{-s}} + \frac{e^{\gamma s}}{1-e^s} \right\} ds
\end{aligned}$$

Therefore, the integrals can be simplified as

$$C_\gamma = \int_0^\infty \frac{\sinh\left\{\left(\frac{1}{2}-\gamma\right)s\right\}}{\sinh\left(\frac{s}{2}\right)}\, ds \tag{A.9}$$

and in particular

$$C_{\frac{1}{2}} = \int_0^\infty \frac{u^{-\frac{1}{2}}du}{1-u} = 0.$$

Note from equations (A.6) and (A.7) that for $0<x<1$, $psi(1-x) - \psi(x) = -\frac{d}{dx}\ln\{\Gamma(x)\Gamma(1-x)\} = \pi\cot(\pi x)$ and therefore, using equation (A.8)

$$\begin{aligned}
\pi\cot(\pi x) &= \int_0^\infty \frac{(e^{-xt} - e^{-(1-x)t})}{1-e^{-t}}\, dt \\
&= \int_0^\infty \frac{\sinh\left\{\left(\frac{1}{2}-x\right)t\right\}}{\sinh\left(\frac{t}{2}\right)}\, dt \quad (0<x<1).
\end{aligned}$$

We conclude that $C_\gamma = \pi\cot(\pi\gamma)$.

The integral in equation (A.9) can be evaluated in other ways, using contour integration, for example.

Some important integrals

1. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \arcsin\frac{x}{a} + C, \quad x^2 \le a^2$

2. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \operatorname{arc\,cosh}\frac{x}{a} + C = \ln|x+\sqrt{x^2-a^2}| + C, \quad a^2 \le x^2$

3. $$\int \frac{1}{a^2-x^2}\,dx = \operatorname{arc\,sinh}\frac{x}{a} + C = \ln|x+\sqrt{a^2-x^2}| + C$$

4. $$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C = -\frac{1}{a}\operatorname{arc\,tanh}\frac{x}{a} + C, \quad a^2 \geq x^2$$

5. $$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C = -\frac{1}{a}\operatorname{arc\,tanh}\frac{x}{a} + C, \quad a^2 \leq x^2$$

6. $$\int x(a+bx)^n\,dx = \frac{(a+bx)^{n+1}}{b^2}\left[\frac{a+bx}{n+2} - \frac{a}{n+1}\right] + C, \quad n \neq -1, -2$$

7. $$\int \frac{1}{x\sqrt{a+bx}}\,dx = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| + C & \text{if } a > 0 \\ \frac{2}{\sqrt{-a}}\arctan\sqrt{\frac{a+bx}{-a}} + C & \text{if } a < 0 \end{cases}$$

8. $$\int \frac{1}{x^n\sqrt{a+bx}}\,dx = -\frac{\sqrt{a+bx}}{a(n-1)x^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{1}{x^{n-1}\sqrt{a+bx}}\,dx$$

9. $$\int \frac{x^n}{\sqrt{a+bx}}\,dx = \frac{2x^n\sqrt{a+bx}}{b(2n+1)} - \frac{2an}{b(2n+1)}\int \frac{x^{n-1}}{\sqrt{a+bx}}dx$$

10. $$\int \frac{\sqrt{a+bx}}{x^n}\,dx = -\frac{(a+bx)^{3/2}}{a(n-1)x^{n-1}} - \frac{b(2n-5)}{2a(n-1)}\int \frac{\sqrt{a+bx}}{x^{n-1}}dx$$

11. $$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\operatorname{arc\,sinh}\frac{x}{a} + C$$

12. $$\int x^2\sqrt{a^2+x^2}\,dx = \frac{x}{8}(a^2+2x^2)\sqrt{a^2+x^2} - \frac{a^4}{8}\operatorname{arc\,sinh}\frac{x}{a} + C$$

13. $$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C, \quad x^2 \leq a^2$$

14. $$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arc\,sec}\left|\frac{x}{a}\right| + C = \frac{1}{a}\arccos\left|\frac{a}{x}\right| + C, \quad a^2 \leq x^2$$

15. $$\int \frac{1}{(a^2+x^2)^2}\,dx = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\arctan\frac{x}{a} + C$$

16. $$\int_0^{\pi/2} \sin^n x\,dx$$
$$= \int_0^{\pi/2} \cos^n x\,dx$$
$$= \begin{cases} \frac{1.3.5\cdots(n-1)}{2.4.6\cdots n}\cdot\frac{\pi}{2}, & \text{if } n \text{ is an even integer} \geq 2 \\ \frac{2.4.6\cdots(n-1)}{3.5.7\cdots n}, & \text{if } n \text{ is an odd integer} \geq 3 \end{cases}$$

17. $$\int \sin^n ax \cos^m ax\, dx$$
$$= \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(m+n)} + \frac{m-1}{m+n}\int \sin^n ax \cos^{m-2} ax\, dx, \quad m \neq -n$$
$$- \frac{\sin^{n-1} ax \cos^{m+1}(ax)}{a(m+n)} + \frac{n-1}{m+n}\int \sin^{n-2} ax \cos^n ax\, dx, \quad n \neq -m$$

18. $$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$$

19. $$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$$

20. $$\int \frac{dx}{b + c \sin ax} = \frac{-2}{a\sqrt{b^2-c^2}} \arctan\left[\sqrt{\frac{b-c}{b+c}} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right] + C, \quad b^2 > c^2$$

21. $$\int \frac{dx}{b + c \sin ax} = \frac{-1}{a\sqrt{c^2-b^2}} \ln\left|\frac{c + b \sin ax + \sqrt{c^2-b^2}\cos ax}{b + c \sin ax}\right| + C, \quad b^2 < c^2$$

22. $$\int \frac{dx}{b + c \cos ax} = \frac{2}{a\sqrt{b^2-c^2}} \arctan\left[\sqrt{\frac{b-c}{b+c}} \tan \frac{ax}{2}\right] + C, \quad b^2 > c^2$$

23. $$\int \frac{dx}{b + c \cos ax} = \frac{1}{a\sqrt{c^2-b^2}} \ln\left|\frac{c + b \cos ax + \sqrt{c^2-b^2}\sin ax}{b + c \cos ax}\right| + C, \quad b^2 < c^2$$

24. $$\int \sec^n ax\, dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax\, dx, \quad n \neq 1$$

25. $$\int \csc^n ax\, dx = \frac{\csc^{n-2} ax \cot ax}{a(n-2)} + \frac{n-2}{n-1}\int \csc^{n-2} ax\, dx, \quad n \neq 1$$

26. $$\int \cot^n ax\, dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int \cot^{n-2} ax\, dx, \quad n \neq 1$$

27. $$\int \sec^n ax \tan ax\, dx = \frac{\sec^n ax}{na} + C, \quad n \neq 0$$

28. $$\int \csc^n ax \cot ax\, dx = -\frac{\csc^n ax}{na} + C, \quad n \neq 0$$

29. $$\int b^{ax} dx = \frac{1}{a}\frac{b^{ax}}{\ln b} + C, \quad b > 0, \quad b \neq 1$$

30. $$\int x^n b^{ax} dx = \frac{x^n b^{ax}}{a \ln b} - \frac{n}{a \ln b}\int x^{n-1} b^{ax} dx, \quad b > 0, \quad b \neq 1$$

31. $\displaystyle\int \ln ax\,dx = x\ln ax - x + C$

32. $\displaystyle\int x^n \ln ax\,dx = \frac{x^{n+1}}{n+1}\ln ax - \frac{x^{n+1}}{(n+1)^2} + C, \quad n \neq 1$

33. $\displaystyle\int \frac{\ln ax}{x}dx = \frac{1}{2}(\ln ax)^2 + C$

34. $\displaystyle\int \frac{1}{x\ln ax}dx = \ln|\ln ax| + C$

35. $\displaystyle\int \sinh^n ax\,dx = \frac{\sinh^{n-1} ax\cosh ax}{na} - \frac{n-1}{n}\int \sinh^{n-2} ax\,dx, \quad n \neq 0$

36. $\displaystyle\int x^n \sinh ax\,dx = \frac{x^n}{a}\cosh ax - \frac{n}{a}\int x^{n-1}\cosh ax\,dx$

37. $\displaystyle\int \cosh^n ax\,dx = \frac{\cosh^{n-1} ax\sinh ax}{na} + \frac{n-1}{n}\int \cosh^{n-2} ax\,dx, \quad n \neq 0$

38. $\displaystyle\int e^{ax}\sinh bx\,dx = \frac{e^{ax}}{2}\left[\frac{e^{bx}}{a+b} - \frac{e^{-bx}}{a-b}\right] + C, \quad a^2 \neq b^2$

39. $\displaystyle\int e^{ax}\cosh bx\,dx = \frac{e^{ax}}{2}\left[\frac{e^{bx}}{a+b} + \frac{e^{-bx}}{a-b}\right] + C, \quad a^2 \neq b^2$

40. $\displaystyle\int \tanh^n ax\,dx = -\frac{\tanh^{n-1} ax}{(n-1)a} + \int \tanh^{n-2} ax\,dx, \quad n \neq 1$

41. $\displaystyle\int \coth^n ax\,dx = -\frac{\coth^{n-1} ax}{(n-1)a} + \int \coth^{n-2} ax\,dx, \quad n \neq 1$

Walli's formula

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \frac{[(m-1)(m-3)\cdots 2 \text{ or } 1][(n-1)(n-3)\cdots 2 \text{ or } 1]}{(m+n)(m+n-2)\cdots 2 \text{ or } 1}\alpha$$

$$\text{where} \qquad \alpha = \begin{cases} \frac{\pi}{2}, & \text{if } m \text{ and } n \text{ are both even} \\ 1, & \text{otherwise.} \end{cases}$$

Appendix B Table of Laplace transforms

Laplace transforms $F(s) = \mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)dt$

Table B.1: General properties of Laplace transforms.

	$F(s)$	$f(t)$
1.	$aF_1(s) + bF_2(s)$	$af_1(t) + bf_2(t)$
2.	$F(as)(a > 0)$	$\frac{1}{a} f(t/a)$
3.	$F(s/a)$	$af(at)$
4.	$F(s-a)$	$e^{at} f(t)$
5.	$F(s+a)$	$e^{-at} f(t)$
6.	$F(as-b)$	$\frac{1}{a} e^{bt/a} f(t/a)$
7.	$\frac{1}{2i}[F(s-ia) - F(s+ia)]$	$f(t)\sin at$
8.	$\frac{1}{2}[F(s-ia) + F(s+ia)]$	$f(t)\cos at$
9.	$\frac{1}{2}[F(s-a) - F(s+a)]$	$f(t)\sinh at$
10.	$\frac{1}{2}[F(s-a) + F(s+a)]$	$f(t)\cosh at$
11.	$e^{-as}F(s)$	$\begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$
12.	$\frac{1}{2} e^{-bs/a} F\left(\frac{s}{a}\right)$ $(a, b > 0)$	$\begin{cases} f(at-b) & t > b/a \\ 0 & t < b/a \end{cases}$
13.	$sF(s) - f(0)$	$f'(t)$

(Continued)

Table B.1: (Continued)

	$F(s)$	$f(t)$
14.	$s^2F(s) - sf(0) - f'(0)$	$f''(t)$
15.	$s^nf(s) - s^{n-1}f(0)$ $- s^{n-2}f'(0) \cdots f^{(n-1)}(0)$	$f^{(n)}(t)$
16.	$F'(s)$	$-tf(t)$
17.	$F''(s)$	$(-t)^2f(t)$
18.	$F^{(n)}(s)$	$(-t)^nf(t)$
19.	$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)d\tau$
20.	$\dfrac{F(s)}{s^n}$	$\int_0^t \cdots \int_0^t f(\tau)d\tau^n$ $= \int_0^t \dfrac{(t-\tau)^{n-1}}{(n-1)!}f(\tau)d\tau$
21.	$F(s)G(s)$	$\int_0^t f(\tau)g(t-\tau)d\tau = f^*g$
22.	$\int_s^\infty F(\sigma)d\sigma$	$\dfrac{f(t)}{t}$
23.	$\int_s^\infty \cdots \int_s^\infty F(\sigma)d\sigma^n$	$\dfrac{f(t)}{t^n}$
24.	$\dfrac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F_1(\sigma)F_2(s-\sigma)d\sigma$	$f_1(t)\cdot f_2(t)$
25.	$\dfrac{\int_0^k e^{-s\tau}f(\tau)d\tau}{1-e^{-ks}}$	$f(t) = f(t+k)$
26.	$\dfrac{F(\sqrt{s})}{s}$	$\dfrac{1}{\sqrt{\pi t}}\int_0^\infty e^{-\tau^2/4t}f(\tau)d\tau$
27.	$\frac{1}{s}F\left(\frac{1}{s}\right)$	$\int_0^\infty J_0(2\sqrt{\tau t})f(\tau)d\tau$
28.	$\frac{1}{s^{n+1}}F\left(\frac{1}{s}\right)$	$t^{n/2}\int_0^\infty \tau^{-n/2}J_n(2\sqrt{\tau t})f(\tau)d\tau$
29.	$\dfrac{F(s+1/s)}{s^2+1}$	$\int_0^t J_0(2\sqrt{\tau(t-\tau)})f(\tau)d\tau$
30.	$\dfrac{1}{2\sqrt{\pi}}\int_0^\infty \tau^{-3/2}e^{-s^2/4\tau}f(\tau)d\tau$	$f(t^2)$
31.	$\dfrac{F(\ln s)}{s\ln s}$	$\int_0^\infty \dfrac{t^\tau f(\tau)}{\Gamma(\tau+1)}d\tau$
32.	$\dfrac{P(s)}{Q(s)}$ *	$\sum_{k=1}^n (P(ak)/Q'(ak))e^{a_kt}$

$*$ $P(s) =$ polynomial of degree less than n,
$Q(s) = (s-a_1)(s-a_2)\ldots(s-a_n), a_1, a_2, \ldots, a_n$ are all distinct.

Table B.2: Special Laplace transforms.

	$F(s)$	$f(t)$
33.	1	$\delta(t)$
34.	$\frac{1}{s}$	$1, u(t)$
35.	$\frac{1}{s^2}$	t
36.	$\frac{1}{s^n}, \quad n=1,2,3,\ldots$	$\frac{t^{n-1}}{(n-1)!}, \quad 0!=1$
37.	$\frac{1}{s^n}, \quad n>0$	$\frac{t^{n-1}}{\Gamma(n)}$
38.	$\frac{1}{s-a}$	e^{at}
39.	$\frac{1}{1+as}$	$\frac{1}{a}e^{-t/a}$
40.	$\frac{1}{(s-a)^n}, \quad n=1,2,3,\ldots$	$\frac{t^{n-1}}{(n-1)!}e^{at}, \quad 0!=1$
41.	$\frac{1}{(s-a)^n}$	$\frac{t^{n-1}}{\Gamma(n)}e^{at}$
42.	$\frac{a}{s^2+a^2}$	$\sin at$
43.	$\frac{s}{s^2+a^2}$	$\cos at$
44.	$\frac{a}{(s-b)^2+a^2}$	$e^{bt}\sin at$
45.	$\frac{s-b}{(s-b)^2+a^2}$	$e^{bt}\cos at$
46.	$\frac{a}{s^2-a^2}$	$\sinh at$
47.	$\frac{s}{s^2-a^2}$	$\cosh at$
48.	$\frac{a}{(s-b)^2-a^2}$	$e^{bt}\sinh at$
49.	$\frac{s-b}{(s-b)^2-a^2}$	$e^{bt}\cosh at$
50.	$\frac{1}{s(s-a)}$	$\frac{1}{a}(e^{at}-1)$
51.	$\frac{1}{s(1+as)}$	$1-e^{-t/a}$
52.	$\frac{1}{(s-a)(s-b)}, \quad a\neq b$	$\frac{e^{at}-e^{bt}}{a-b}$

(Continued)

Table B.2: (Continued)

	$F(s)$	$f(t)$
53.	$\dfrac{1}{(1+as)(1+bs)}, \quad a \neq b$	$\dfrac{e^{-t/a} - e^{-t/b}}{a-b}$
54.	$\dfrac{1}{s^2+2bs+(b^2+\omega^2)}$	$\dfrac{1}{\omega} e^{-bt} \sin \omega t$
55.	$\dfrac{s}{(s-a)(s-b)}$	$\dfrac{be^{bt} - ae^{at}}{b-a}$
56.	$\dfrac{1}{(s^2+a^2)^2}$	$\dfrac{\sin at - at \cos at}{2a^3}$
57.	$\dfrac{s}{(s^2+a^2)^2}$	$\dfrac{t \sin at}{2a}$
58.	$\dfrac{s^2}{(s^2+a^2)^2}$	$\dfrac{\sin at + at \cos at}{2a}$
59.	$\dfrac{s^3}{(s^2+a^2)^2}$	$\cos at - \dfrac{at}{2} \sin at$
60.	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$	$t \cos at$
61.	$\dfrac{1}{(s^2-a^2)^2}$	$\dfrac{at \cosh at - \sinh at}{2a^3}$
62.	$\dfrac{s}{(s^2-a^2)^2}$	$\dfrac{t \sinh at}{2a}$
63.	$\dfrac{s^2}{(s^2-a^2)^2}$	$\dfrac{\sinh at + at \cosh at}{2a}$
64.	$\dfrac{s^3}{(s^2-a^2)^2}$	$\cosh at + \dfrac{at}{2} \sinh at$
65.	$\dfrac{s^2+a^2}{(s^2+a^2)^2}$	$t \cosh at$
66.	$\dfrac{1}{(s^2+a^2)^3}$	$\dfrac{(3-a^2t^2) \sin at - 3at \cos at}{8a^5}$
67.	$\dfrac{s}{(s^2+a^2)^3}$	$\dfrac{t \sin at - at^2 \cos at}{8a^3}$
68.	$\dfrac{s^2}{(s^2+a^2)^3}$	$\dfrac{(1+a^2t^2) \sin at - at \cos at}{8a^3}$
69.	$\dfrac{s^3}{(s^2+a^2)^3}$	$\dfrac{3t \sin at + at^2 \cos at}{8a}$
70.	$\dfrac{s^4}{(s^2+a^2)^3}$	$\dfrac{(3-a^2t^2) \sin at + 5at \cos at}{8a}$

(Continued)

Table B.2: (Continued)

	$F(s)$	$f(t)$
71.	$\dfrac{s^5}{(s^2+a^2)^3}$	$\dfrac{(8-a^2t^2)\cos at - 7at\sin at}{8}$
72.	$\dfrac{3s^2-a^2}{(s^2+a^2)^3}$	$\dfrac{t^2\sin at}{2a}$
73.	$\dfrac{s^3-3a^2s}{(s^2+a^2)^3}$	$\dfrac{1}{2}t^2\cos at$
74.	$\dfrac{s^3-a^2s}{(s^2+a^2)^4}$	$\dfrac{t^3\sin at}{24a}$
75.	$\dfrac{1}{(s^2-a^2)^3}$	$\dfrac{(3+a^2t^2)\sinh at - 3at\cosh at}{8a^5}$
76.	$\dfrac{s}{(s^2-a^2)^3}$	$\dfrac{at^2\cosh at - t\sinh at}{8a^3}$
77.	$\dfrac{s^2}{(s^2-a^2)^3}$	$\dfrac{at\cosh at + (a^2t^2-1)\sinh at}{8a^3}$
78.	$\dfrac{s^3}{(s^2-a^2)^3}$	$\dfrac{3t\sinh at + at^2\cosh at}{8a}$
79.	$\dfrac{s^4}{(s^2-a^2)^3}$	$\dfrac{(3+a^2t^2)\sinh at + 5at\cosh at}{8a}$
80.	$\dfrac{s^5}{(s^2-a^2)^3}$	$\dfrac{(8+a^2t^2)\cosh at + 7at\sinh at}{8}$
81.	$\dfrac{3s^2+a^2}{(s^2-a^2)^3}$	$\dfrac{t^2\sinh at}{2a}$
82.	$\dfrac{s^2+2a^2}{s(s^2+4a^2)}$	$\cos^2 at$
83.	$\dfrac{s^2-2a^2}{s(s^2-4a^2)}$	$\cosh^2 at$
84.	$\dfrac{2a^2}{s(s^2+4a^2)}$	$\sin^2 at$
85.	$\dfrac{2a^2}{s(s^2-4a^2)}$	$\sinh^2 at$
86.	$\dfrac{1}{s^3+a^3}$	$\dfrac{e^{at/2}}{3a^2}\left\{3\sin\dfrac{\sqrt{3}at}{2} - \cos\dfrac{\sqrt{3}at}{2} + e^{-3at/2}\right\}$
87.	$\dfrac{s}{s^3+a^3}$	$\dfrac{e^{at/2}}{3a}\left\{\cos\dfrac{\sqrt{3}at}{2} + \sqrt{3}\sin\dfrac{\sqrt{3}at}{2} - e^{-3at/2}\right\}$

(Continued)

Table B.2: (Continued)

	$F(s)$	$f(t)$
88.	$\dfrac{s^2}{s^3+a^3}$	$\dfrac{1}{3}\left\{e^{-at}+2e^{at/2}\cos\dfrac{\sqrt{3}at}{2}\right\}$
89.	$\dfrac{1}{s^3-a^3}$	$\dfrac{e^{-at/2}}{3a^2}\left\{e^{3at/2}-\cos\dfrac{\sqrt{3}at}{2}-\sqrt{3}\sin\dfrac{\sqrt{3}}{2}at\right\}$
90.	$\dfrac{s}{s^3-a^3}$	$\dfrac{e^{-at/2}}{3a}\left\{\sqrt{3}\sin\dfrac{\sqrt{3}at}{2}-\cos\dfrac{\sqrt{3}at}{2}+e^{3at/2}\right\}$
91.	$\dfrac{s^2}{s^3-a^3}$	$\dfrac{1}{3}\left\{e^{at}+2e^{-at/2}\cos\dfrac{\sqrt{3}at}{2}\right\}$
92.	$\dfrac{1}{s^4+4a^4}$	$\dfrac{1}{4a^3}(\sin at\cosh at-\cos at\sinh at)$
93.	$\dfrac{s}{s^4+4a^4}$	$\dfrac{\sin at\sinh at}{2a^2}$
94.	$\dfrac{s^2}{s^4+4a^4}$	$\dfrac{1}{2a}(\sin at\cosh at+\cos at\sinh at)$
95.	$\dfrac{s^3}{s^4+4a^4}$	$\cos at\cosh at$
96.	$\dfrac{1}{s^4-a^4}$	$\dfrac{1}{2a^3}(\sinh at-\sin at)$
97.	$\dfrac{s}{s^4-a^4}$	$\dfrac{1}{2a^2}(\cosh at-\cos at)$
98.	$\dfrac{s^2}{s^4-a^4}$	$\dfrac{1}{2a}(\sinh at+\sin at)$
99.	$\dfrac{s^3}{s^4-a^4}$	$\dfrac{1}{2}(\cosh at+\cos at)$
100.	$\dfrac{1}{\sqrt{s}}$	$\dfrac{1}{\sqrt{\pi t}}$
101.	$\dfrac{1}{s\sqrt{s}}$	$2\sqrt{t/\pi}$
102.	$\dfrac{s+a}{s\sqrt{s}}$	$\dfrac{1+2at}{\sqrt{\pi t}}$
103.	$\dfrac{1}{\sqrt{s+a}}$	$\dfrac{e^{-at}}{\sqrt{\pi t}}$
104.	$\sqrt{s-a}-\sqrt{s-b}$	$\dfrac{e^{bt}-e^{at}}{2t\sqrt{\pi t}}$

(Continued)

Table B.2: (Continued)

	$F(s)$	$f(t)$
105.	$\dfrac{1}{\sqrt{s+a}+\sqrt{s+b}}$	$\dfrac{e^{-bt}-e^{-at}}{2(b-a)\sqrt{\pi t^3}}$
106.	$\dfrac{1}{s\sqrt{s+a}}$	$\dfrac{erf\sqrt{at}}{\sqrt{a}}$
107.	$\dfrac{1}{\sqrt{s}(s-a)}$	$\dfrac{e^{at}erf\sqrt{at}}{\sqrt{a}}$
108.	$\dfrac{1}{\sqrt{s-a}+b}$	$e^{at}\left\{\dfrac{1}{\sqrt{\pi t}}-be^{b^2t}er\{fc(b\sqrt{t})\}\right.$
109.	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
110.	$\dfrac{1}{\sqrt{s^2-a^2}}$	$I_0(at)$
111.	$\dfrac{(\sqrt{s^2+a^2}-s)^n}{\sqrt{s^2+a^2}}, \quad n>-1$	$a^nJ_n(at)$
112.	$\dfrac{(s-\sqrt{s^2-a^2})^n}{\sqrt{s^2-a^2}}, \quad n>-1$	$a^nI_n(at)$
113.	$\dfrac{e^{b(s-\sqrt{s^2+a^2})}}{\sqrt{s^2+a^2}}$	$J_0(a\sqrt{t(t+2b)})$
114.	$\dfrac{e^{-b\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}$	$\begin{cases} J_0(a\sqrt{t^2-b^2}), & t>b \\ 0, & t<b \end{cases}$
115.	$\dfrac{1}{s^n\sqrt{s}}$	$\dfrac{4^n n!}{(2n)!\sqrt{\pi}}t^{n-1/2}$
116.	$\dfrac{1}{(s^2+a^2)^{3/2}}$	$\dfrac{t\,J_1(at)}{a}$
117.	$\dfrac{s}{(s^2+a^2)^{3/2}}$	$t\,J_0(at)$
118.	$\dfrac{s^2}{(s^2+a^2)^{3/2}}$	$J_0(at)-at\,J_1(at)$
119.	$\dfrac{1}{(s^2-a^2)^{3/2}}$	$\dfrac{t\,I_1(at)}{a}$
120.	$\dfrac{s}{(s^2-a^2)^{3/2}}$	$t\,I_0(at)$
121.	$\dfrac{s^2}{(s^2-a^2)^{3/2}}$	$I_0(at)+at\,I_1(at)$

(Continued)

Table B.2: (Continued)

	$F(s)$	$f(t)$
122.	$\dfrac{\ln s}{s}$	$-(\ln t+\gamma)$, $\gamma =$ Euler's constant, $= 0.5772156$
123.	$-\dfrac{(\gamma+\ln s)}{s}$	$\ln t$
124.	$-\sqrt{\dfrac{\pi}{s}}(\ln 4s+\gamma)$	$\dfrac{\ln t}{\sqrt{t}}$
125.	$\dfrac{(\ln s)^2}{s}$	$(\ln t+\gamma)^2-\dfrac{\pi^2}{b}$
126.	$\dfrac{1}{\ln s}$	$\displaystyle\int_0^{\infty}\frac{t^{\tau-1}}{\Gamma(\tau)}d\tau$
127.	$\dfrac{\pi^2}{6s}+\dfrac{\gamma+\ln(s)^2}{s}$	$\ln^2 t$
128.	$\dfrac{\Gamma'(n+1)-\Gamma(n+1)\ln s}{s^{n+1}}, \quad n>-1$	$t^n \ln t$
129.	$\ln\left(\dfrac{s-a}{s}\right)$	$\dfrac{1-e^{at}}{t}$
130.	$\ln\left(\dfrac{s-a}{s-b}\right)$	$\dfrac{e^{bt}-e^{at}}{t}$
131.	$\dfrac{\ln[(s^2+a^2)/a^2]}{2s}$	$C_i(at)=\displaystyle\int_{at}^{\infty}\frac{\cos\tau}{\tau}d\tau$
132.	$\dfrac{\ln[(s+a)/a]}{s}$	$E_i(at)=\displaystyle\int_{at}^{\infty}\frac{e^{-\tau}}{\tau}d\tau$
133.	$\ln\dfrac{s^2+a^2}{s^2+b^2}$	$\dfrac{2(\cos at-\cos bt)}{t}$
134.	e^{-as}	$\delta(t-a)$
135.	$\dfrac{e^{-as}}{s}$	$u(t-a)$
136.	$\dfrac{e^{-a^2/4s}}{s}$	$J_0(a\sqrt{t})$
137.	$\tan^{-1}\left(\dfrac{a}{s}\right)$	$\dfrac{\sin at}{t}$
138.	$\dfrac{\tan^{-1}(a/s)}{s}$	$S_i(at)=\displaystyle\int_0^{at}\frac{\sin\tau}{\tau}d\tau$

(Continued)

Table B.2: (Continued)

	$F(s)$	$f(t)$
139.	$\frac{e^{a/s}}{\sqrt{s}} erfc(\sqrt{a/s})$	$\frac{e^{-2\sqrt{at}}}{\sqrt{\pi t}}$
140.	$e^{s^2/4a^2} erfc(s/2a)$	$\frac{2a}{\sqrt{\pi}} e^{-a^2 t^2}$
141.	$\frac{e^{s^2/4a^2} erfc(s/2a)}{s}$	$erf(at)$
142.	$\frac{e^{as} erfc\sqrt{as}}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi(t+a)}}$
143.	$e^{as} E_i(as)$	$\frac{1}{t+a}$
144.	$\frac{1}{\sqrt{\pi s}} e^{-a^2/4s}$	$\frac{\cos a\sqrt{t}}{\pi\sqrt{t}}$
145.	$\frac{a}{2\sqrt{\pi} s^{3/2}} e^{-a^2/4s}$	$\frac{\sin a\sqrt{t}}{\pi}$
146.	$\frac{e^{1/s}}{\sqrt{s}}$	$\frac{\cosh 2\sqrt{t}}{\sqrt{t}}$
147.	$\frac{e^{1/s}}{s\sqrt{s}}$	$\frac{\sinh 2\sqrt{t}}{\sqrt{\pi}}$
148.	$e^{-a\sqrt{s}}$	$\frac{a}{2\sqrt{\pi} t^{3/2}} e^{-a^2/4t}$
149.	$\frac{1-e^{-a\sqrt{s}}}{s}$	$erf\left(\frac{a}{2\sqrt{t}}\right)$
149a.	$\frac{e^{-a\sqrt{s}}}{s}$	$erfc\left(\frac{a}{2\sqrt{t}}\right)$
150.	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}} e^{-a^2/4t}$
151.	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s}+b)}$	$e^{b(bt+a)} erfc\left(b\sqrt{t}+\frac{a}{2\sqrt{t}}\right)$
152.	$\frac{e^{-a\sqrt{s}}}{s(b+\sqrt{s})}$	$\frac{1}{b} erfc\left(\frac{a}{2\sqrt{t}}\right) - \frac{1}{b} e^{b^2 t+ab} erfc\left(b\sqrt{t}+\frac{a}{2\sqrt{t}}\right)$
152a.	$\frac{e^{-a\sqrt{s}}}{s\sqrt{s}}$	$\frac{2\sqrt{t}}{\sqrt{\pi}} e^{-a^2/4t} - a\,erfc\left(\frac{a}{2\sqrt{t}}\right)$

(Continued)

Table B.2: (Continued)

	$F(s)$	$f(t)$
153.	$e^{-x\sqrt{as^2+bs+c}} - e^{-(bx/2\sqrt{a})}e^{-\sqrt{a}xs}$	$\begin{cases} 0 & 0 \le t \le \sqrt{a}x \\ \sqrt{\dfrac{b^2-4ca}{4a}} & x\,e^{-(b/2a)t} \\ \dfrac{I_1\left(\dfrac{\sqrt{b^2-4ca}}{2a}\sqrt{t^2-ax^2}\right)}{\sqrt{t^2-ax^2}}, & t \ge \sqrt{a}x \end{cases}$
154.	$\dfrac{1}{\sqrt{s}}\sin\dfrac{a}{s}$	$\dfrac{\sinh\sqrt{2at}\sin\sqrt{2at}}{\sqrt{\pi t}}$
155.	$\dfrac{1}{s\sqrt{s}}\sin\dfrac{a}{s}$	$\dfrac{\cosh\sqrt{2at}\sin\sqrt{2at}}{\sqrt{a\pi}}$
156.	$\dfrac{1}{s\sqrt{s}}\cos\dfrac{a}{s}$	$\dfrac{\cosh\sqrt{2at}\cos\sqrt{2at}}{\sqrt{\pi t}}$
157.	$\dfrac{1}{s\sqrt{s}}\cos\dfrac{a}{s}$	$\dfrac{\sinh\sqrt{2at}\cos\sqrt{2at}}{\sqrt{a\pi}}$
158.	$\tan^{-1}\dfrac{s^2-as-b}{ab}$	$\dfrac{e^{at}-1}{t}\sin bt$
159.	$\tan^{-1}\dfrac{2as}{s^2-a^2+b^2}$	$\dfrac{2}{t}\sin at\cos bt$
160.	$\dfrac{\sqrt{\pi}}{2}e^{(s/2)^2}erfc\left(\dfrac{s}{2}\right)$	e^{-t^2}
161.	$\dfrac{\sinh sx}{s\sinh sa}$	$\dfrac{x}{a}+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin\dfrac{n\pi x}{a}\cos\dfrac{n\pi t}{a}$
162.	$\dfrac{\sinh sx}{s\cosh sa}$	$\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}\sin\dfrac{(2n-1)\pi x}{2a}\sin\dfrac{(2n-1)\pi t}{2a}$
163.	$\dfrac{\cosh sx}{s\sinh sa}$	$\dfrac{t}{a}+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\cos\dfrac{n\pi x}{a}\sin\dfrac{n\pi t}{a}$
164.	$\dfrac{\cosh sx}{s\cosh sa}$	$1+\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}\cos\dfrac{(2n-1)\pi x}{2a}\cos\dfrac{(2n-1)\pi t}{2a}$
165.	$\dfrac{\sinh sx}{s^2\sinh sa}$	$\dfrac{xt}{a}+\dfrac{2a}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\sin\dfrac{n\pi x}{a}\sin\dfrac{n\pi t}{a}$
166.	$\dfrac{\sinh sx}{s^2\cosh sa}$	$x+\dfrac{8a}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(2n-1)^2}\sin\dfrac{(2n-1)\pi x}{2a}\cos\dfrac{(2n-1)\pi t}{2a}$

(Continued)

Table B.2: (Continued)

	$F(s)$	$f(t)$
167.	$\dfrac{\cosh sx}{s^2 \sinh sa}$	$\dfrac{t^2}{2a} + \dfrac{2a}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos \dfrac{n\pi x}{a} \left(1 - \cos \dfrac{n\pi t}{a}\right)$
168.	$\dfrac{\cosh sx}{s^2 \cosh sa}$	$t + \dfrac{8a}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^n}{(2n-1)^2} \cos \dfrac{(2n-1)\pi x}{2a} \sin \dfrac{(2n-1)\pi t}{2a}$
169.	$\dfrac{\cosh sx}{s^3 \cosh sa}$	$\dfrac{1}{2}(t^2 + x^2 - a^2) - \dfrac{16a^2}{\pi^3} \sum_{n=1}^{\infty} \dfrac{(-1)^n}{(2n-1)^3}$ $\times \cos \dfrac{(2n-1)\pi x}{2a} \cos \dfrac{(2n-1)\pi t}{2a}$
170.	$\dfrac{\sinh x\sqrt{s}}{\sinh a\sqrt{s}}$	$\dfrac{2\pi}{a^2} \sum_{n=1}^{\infty} (-1)^n n e^{-n^2\pi^2 t/a^2} \sin \dfrac{n\pi x}{a}$
171.	$\dfrac{\cosh x\sqrt{s}}{\cosh a\sqrt{s}}$	$\dfrac{\pi}{a^2} \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1) e^{-(2n-1)^2\pi^2 t/4a^2}$ $\times \cos \dfrac{(2n-1)\pi x}{2a}$
172.	$\dfrac{\sinh x\sqrt{s}}{\sqrt{s}\cosh a\sqrt{s}}$	$\dfrac{2}{a} \sum_{n=1}^{\infty} (-1)^{n-1} e^{-(2n-1)^2\pi^2 t/4a^2} \sin \dfrac{(2n-1)\pi x}{2a}$
173.	$\dfrac{\cosh x\sqrt{s}}{\sqrt{s}\sinh a\sqrt{s}}$	$\dfrac{1}{a} + \dfrac{2}{a} \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2 t/a^2} \cos \dfrac{n\pi x}{a}$
174.	$\dfrac{\sinh x\sqrt{s}}{s \sinh a\sqrt{s}}$	$\dfrac{x}{a} + \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} e^{-n^2\pi^2 t/a^2} \sin \dfrac{n\pi x}{a}$
175.	$\dfrac{\cosh x\sqrt{s}}{s \cosh a\sqrt{s}}$	$1 + \dfrac{4}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^n}{2n-1} e^{-(2n-1)^2\pi^2 t/4a^2} \cos \dfrac{(2n-1)\pi x}{2a}$
176.	$\dfrac{\sinh x\sqrt{s}}{s^2 \sinh a\sqrt{s}}$	$\dfrac{xt}{a} + \dfrac{2a^2}{\pi^3} \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^3} (1 - e^{-n^2\pi^2 t/a^2}) \sin \dfrac{n\pi x}{a}$
177.	$\dfrac{\cosh x\sqrt{s}}{s^2 \cosh a\sqrt{s}}$	$\dfrac{1}{2}(x^2 - a^2) + t - \dfrac{16a^2}{\pi^3} \sum_{n=1}^{\infty}$ $\times \dfrac{(-1)^n}{(2n-1)^3} e^{-(2n-1)^2\pi^2 t/4a^2} \cos \dfrac{(2n-1)\pi x}{2a}$
178.	$\dfrac{1}{as^2} \tanh \left(\dfrac{as}{2}\right)$	Triangular wave function
179.	$\dfrac{1}{s} \tanh \left(\dfrac{as}{2}\right)$	Square wave function
180.	$\dfrac{\pi a}{a^2 s^2 + \pi^2} \coth \left(\dfrac{as}{2}\right)$	Rectified sine wave function
181.	$\dfrac{\pi a}{(a^2 s^2 + \pi^2)(1 - e^{-as})}$	Half-rectified sine wave function

(Continued)

Table B.2: (Continued)

	$F(s)$	$f(t)$
182.	$\dfrac{1}{as^2} - \dfrac{1}{s(1-e^{as})}$	Saw tooth wave function
183.	$\dfrac{e^{-as}}{s}$	Heaviside's unit function $u(t-a)$
184.	$\dfrac{e^{-as}(1-e^{-\epsilon s})}{s}$	Pulse function
185.	$\dfrac{1}{s(1-e^{-as})}$	Step function
186.	$\dfrac{e^{-s}+e^{-2s}}{s(1-e^{-s})^2}$	$f(t)=n^2, n \leq t < n+1, n=0,1,2,\ldots$
187.	$\dfrac{1-e^{-s}}{s(1-re^{-s})}$	$f(t)=r^n, n \leq t < n+1, n=0,1,2,\ldots$
188.	$\dfrac{a(1+e^{-as})}{a^2s^2+\pi^2}$	$f(t)=\begin{cases} \sin(\pi t/a) & 0 \leq t \leq a \\ 0 & t > a \end{cases}$

Table B.3: Special functions.

189.	Gamma function	$\Gamma(n) = \int_0^\infty u^{n-1}e^{-u}du, \quad n > 0$
190.	Beta function	$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}du = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \ m,n > 0$
191.	Bessel function	$J_n(x) = \dfrac{x^n}{2^n\Gamma(n+1)}\left\{1 - \dfrac{x^2}{2(2n+2)} + \dfrac{x^4}{2.4(2n+2)(2n+4)} - \cdots\right\}$
192.	Modified Bessel function	$I_n(x) = i^{-n}J_n(ix) = \dfrac{x^n}{2^n\Gamma(n+1)}\left\{1 + \dfrac{x^2}{2(2n+2)} + \cdots\right\}$
193.	Error function	$erf(t) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^t e^{-u^2}du$
194.	Complementary error function	$erfc(t) = 1 - erf(t) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_t^\infty e^{-u^2}du$

(Continued)

Table B.3: (Continued)

195.	Exponential integral	$Ei(t) = \int_t^\infty \frac{e^{-u}}{u} du$
196.	Sine integral	$Si(t) = \int_0^t \frac{\sin u}{u} du$
197.	Cosine integral	$Ci(t) = \int_t^\infty \frac{\cos u}{u} du$
198.	Fresnel sine integral	$S(t) = \int_0^t \sin u^2 du$
199.	Fresnel cosine integral	$C(t) = \int_0^t \cos u^2 du$
200.	Laguerre polynomials	$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n}(t^n e^{-t}), \quad n = 0, 1, 2, \ldots$

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Appendix C Specialized Laplace inverses

Laplace transforms $F(s) = \mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)dt$

Note that we write $q = \sqrt{s/\alpha}$, α and x are always real and positive, k and h are unrestricted.

Table C.1: Advanced Laplace transforms.

	$F(s)$	$f(t)$
1.	e^{-qx}	$\dfrac{x}{2\sqrt{(\pi\alpha t^3)}} e^{-x^2/4\alpha t}$
2.	$\dfrac{e^{-qx}}{q}$	$\sqrt{\left(\dfrac{\alpha}{\pi t}\right)} e^{-x^2/4\alpha t}$
3.	$\dfrac{e^{-qx}}{s}$	$erfc \dfrac{x}{2\sqrt{(\alpha t)}}$
4.	$\dfrac{e^{-qx}}{sq}$	$2\sqrt{\left(\dfrac{\alpha t}{\pi}\right)} e^{-x^2/4\alpha t} - x erfc \dfrac{x}{2\sqrt{(\alpha t)}}$
5.	$\dfrac{e^{-qx}}{s^2}$	$\left\{t + \dfrac{x^2}{2\alpha}\right\} erfc \dfrac{x}{2\sqrt{(\alpha t)}} - x\left(\dfrac{t}{\pi\alpha}\right)^{\frac{1}{2}} e^{-x^2/4\alpha t}$
6.	$e^{-qx}, s^{1+\frac{1}{2}n}$, $n = 0, 1, 2, \ldots$	$(4t)^{\frac{1}{2}n} i^n erfc \dfrac{x}{2\sqrt{\alpha t}}$
7.	$\dfrac{e^{-qx}}{q+h}$	$\sqrt{\left(\dfrac{\alpha}{\pi t}\right)} e^{-x^2/4\alpha t} - h\alpha e^{hx+\alpha h^2 t} \times erfc \left\{\dfrac{x}{2\sqrt{\alpha t}} + h\sqrt{(\alpha t)}\right\}$
8.	$\dfrac{e^{-qx}}{q(q+h)}$	$\alpha e^{hx+\alpha t h^2} erfc \left\{\dfrac{x}{2\sqrt{(\alpha t)}} + h\sqrt{(\alpha t)}\right\}$

(Continued)

Table C.1: (Continued)

	$F(s)$	$f(t)$
9.	$\dfrac{e^{-qx}}{s(q+h)}$	$\dfrac{1}{h}erfc\dfrac{x}{2\sqrt{(\alpha t)}} - \dfrac{1}{h}e^{hx+\alpha th^2} \times erfc\left\{\dfrac{x}{2\sqrt{(\alpha t)}} + h\sqrt{(\alpha t)}\right\}$
10.	$\dfrac{e^{-qx}}{sq(q+h)}$	$\dfrac{2}{h}\sqrt{\left(\dfrac{\alpha t}{\pi}\right)}e^{-x^2/4\alpha t} - \dfrac{(1+hx)}{h^2}erfc\dfrac{x}{2\sqrt{(\alpha t)}}$ $+\dfrac{1}{h^2}e^{hx+\alpha th^2}erfc\left\{\dfrac{x}{2\sqrt{(\alpha t)}} + h\sqrt{(\alpha t)}\right\}$
11.	$\dfrac{e^{-qx}}{q^{n+1}(q+h)}$	$\dfrac{\alpha}{(-h)^n}e^{hx+\alpha th^2}erfc\left\{\dfrac{x}{2\sqrt{(\alpha t)}} + \sqrt{(\alpha t)}\right\}$ $-\dfrac{\alpha}{(-h)^n}\sum_{r=0}^{n-1}\left[-2h\sqrt{(\alpha t)}\right]^r i^r erfc\dfrac{x}{2\sqrt{(\alpha t)}}$
12.	$\dfrac{e^{-qx}}{(q+h)^2}$	$-2h\sqrt{\left(\dfrac{\alpha^3 t}{\pi}\right)}e^{-x^2/4\alpha t} + \alpha(1+hx+2h^2\alpha t)e^{hx+\alpha th^2}$ $\times erfc\left\{\dfrac{x}{2\sqrt{(\alpha t)}} + h\sqrt{(\alpha t)}\right\}$
13.	$\dfrac{e^{-qx}}{s(q+h)^2}$	$\dfrac{1}{h^2}erfc\dfrac{x}{2\sqrt{(\alpha t)}} - \dfrac{2}{h}\sqrt{\left(\dfrac{\alpha t}{\pi}\right)}e^{-x^2/4\alpha t}$ $-\dfrac{1}{h^2}\{1-hx-2h^2\alpha t\}e^{hx+\alpha th^2}$ $\times erfc\left\{\dfrac{x}{2\sqrt{(\alpha t)}} + h\sqrt{(\alpha t)}\right\}$
14.	$\dfrac{e^{-qx}}{s-\alpha}$	$\dfrac{1}{2}e^{kt}\left\{e^{-x\sqrt{(k/\alpha)}}erfc\left[\dfrac{x}{2\sqrt{(\alpha t)}} - \sqrt{kt}\right]\right.$ $\left.+e^{x\sqrt{(k/\alpha)}}erfc\left[\dfrac{x}{2\sqrt{(\alpha t)}} + \sqrt{kt}\right]\right\}$
15.	$\dfrac{1}{s^{\frac{3}{4}}}e^{-qx}$	$\dfrac{1}{\pi}\sqrt{\left(\dfrac{x}{2t\sqrt{\alpha}}\right)}e^{-x^2/8\alpha t}K_{\frac{1}{4}}\left(\dfrac{x^2}{8\alpha t}\right)$
16.	$\dfrac{1}{s^{\frac{1}{2}}}K_{2v}(qx)$	$\dfrac{1}{2\sqrt{\pi t}}e^{-x^2/8\alpha t}K_v\left(\dfrac{x^2}{8\alpha t}\right)$

(Continued)

Table C.1: (Continued)

	$F(s)$	$f(t)$
17.	$I_v(qx')K_v(qx)$, $x > x'$ $I_v(qx)K_v(qx')$, $x < x'$	$\frac{1}{2t}e^{-(x^2+x'^2)/4\alpha t}I_v\left(\frac{xx'}{2\alpha t}\right), \quad v \geq 0$
18.	$K_0(qx)$	$\frac{1}{2t}e^{-x^2/4\alpha t}$
19.	$\frac{1}{s}e^{x/s}$	$I_0[2\sqrt{(xt)}]$
20.	$\frac{e^{-qx}}{(s-\alpha)^2}$	$\frac{1}{2}e^{kt}\left\{\left(t-\frac{x}{2\sqrt{(k\alpha)}}\right)\right.$ $\times e^{-x\sqrt{(k/\alpha)}}erfc\left[\frac{x}{2\sqrt{(\alpha t)}}-\sqrt{(kt)}\right]$ $+\left(t+\frac{x}{2\sqrt{(k\alpha)}}\right)e^{x\sqrt{(k/\alpha)}}erfc\left[\frac{x}{2\sqrt{(\alpha t)}}+\sqrt{(kt)}\right]$
21.	$\frac{e^{-qx}}{q(s-k)}$	$\frac{1}{2}e^{kt}\sqrt{\left(\frac{\alpha}{k}\right)}\left\{e^{-x\sqrt{(k/\alpha)}}erfc\left[\frac{x}{2\sqrt{(\alpha t)}}-\sqrt{(kt)}\right]\right.$ $\left.-e^{x\sqrt{(k/\alpha)}}erfc\left[\frac{x}{2\sqrt{(\alpha t)}}+\sqrt{(kt)}\right]\right\}$
22.	$\frac{e^{-qx}}{(s-k)(q+h)}$, $k \neq \alpha h^2$	$\frac{1}{2}e^{kt}\left\{\frac{\alpha^{\frac{1}{2}}}{h\alpha^{\frac{1}{2}}+k^{\frac{1}{2}}}e^{-x\sqrt{(k/\alpha)}}erfc\left[\frac{x}{2\sqrt{(\alpha t)}}-\sqrt{(kt)}\right]\right.$ $+\frac{\alpha^{\frac{1}{2}}}{h\alpha^{\frac{1}{2}}-k^{\frac{1}{2}}}e^{x\sqrt{(k/\alpha)}}erfc\left[\frac{x}{2\sqrt{(\alpha t)}}+\sqrt{(kt)}\right]$ $-\frac{h\alpha}{h^2\alpha-k}e^{hx+h^2\alpha t}erfc\left[\frac{x}{2\sqrt{(\alpha t)}}+h\sqrt{(\alpha t)}\right]$
23.	$\frac{1}{s}\ln s$	$-\ln(Ct), \quad \ln C = \gamma = 0.5772\ldots$
24.	$s^{\frac{1}{2}v}K_v(x\sqrt{s})$	$\frac{x^v}{(2t)^{v+1}}e^{-x^2/4t}$

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Answers to some selected exercises

Chapter 1

1(a). Nonhomogeneous linear Fredholm integral equation.

1(b). Nonhomogeneous nonlinear Volterra integral equation.

1(c). Nonhomogeneous linear Fredholm integral equation.

1(d). Nonhomogeneous nonlinear Volterra integral equation.

2(a). Nonlinear Volterra integro-differential equation.

2(b). Nonlinear Volterra integro-differential equation.

2(c). Linear Fredholm integro-differential equation.

3(a). $u(x) = 4 + \int_0^x u^2(t)dt.$

3(b). $u'(x) = 1 + 4\int_0^x tu^2(t)dt, \quad u(0) = 2.$

3(c). $u'(x) = 1 + 2\int_0^x tu^2(t)dt, \quad u(0) = 0.$

5(a). $u''(x) - u(x) = \cos x, \quad u(0) = -1, u'(0) = 1.$

5(b). $u'''(x) - 4u(x) = 24x, \quad u(0) = 0, u'(0) = 0, u''(0) = 2.$

5(c). $u^{iv}(x) - u(x) = 0, \quad u(0) = u'(0) = 0, u''(0) = 2, u'''(0) = 0.$

6(a). $u(x) = -11 - 6x - \int_0^x (5 + 6(x-t))u(t)dt.$

6(b). $u(x) = \sin x - \int_0^x (x-t)u(t)dt.$

6(c). $u(x) = -3x - 4\int_0^x (x-t)u(t)dt.$

6(d). $u(x) = 2e^x - 1 - x - \int_0^x (x-t)u(t)dt.$

7(a). $u(x) = \sin x + \int_0^1 K(x,t)u(t)dt$, where the kernel $K(x,t)$ is defined by

$$K(x,t) = \begin{cases} 4t(1-x) & 0 \le t \le x \\ 4x(1-t) & x \le t \le 1. \end{cases}$$

7(b). $u(x) = 1 + \int_0^1 K(x,t)u(t)dt$, where the kernel $K(x,t)$ is defined by

$$K(x,t) = \begin{cases} 2xt(1-x) & 0 \le t \le x \\ 2x^2(1-t) & x \le t \le 1. \end{cases}$$

7(c). $u(x) = (2x-1) + \int_0^1 K(x,t)u(t)dt$, where the kernel $K(x,t)$ is defined by

$$K(x,t) = \begin{cases} t(1-x) & 0 \le t \le x \\ x(1-t) & x \le t \le 1. \end{cases}$$

7(d). $u(x) = (x-1) + \int_0^1 K(x,t)u(t)dt$, where the kernel $K(x,t)$ is defined by

$$K(x,t) = \begin{cases} t & 0 \le t \le x \\ x & x \le t \le 1. \end{cases}$$

Chapter 2

1(a). $u(x) = 1 + 2x$.

1(b). $u(x) = e^x$.

1(c). $u(x) = 2\cosh x - 1$.

1(d). $u(x) = \sec^2 x$.

1(e). $u(x) = x^3$.

2(a). $u(x) = 2x + 3x^2$.

2(b). $u(x) = e^{-x}$.

2(c). $u(x) = \cos x - \sin x$.

2(d). $u(x) = \sin x$.

3(a). $u(x) = e^{-x}$.

3(b). $u(x) = 2x$.

3(c). $u(x) = \sinh x$.

3(d). $u(x) = \sec^2 x$.

4. $u(x) = \cos x$.

5. $u(x) = \frac{60}{7}e^{6x} + \frac{3}{7}e^{-x}$.

6. $u(x) = -\frac{1}{2}x\sin x - \cos x - \sin x$.

8. Hints: Take the Laplace transform of both sides and determine the inversion of the resulting transform. Here, the constants are related to: $\lambda+\nu=c$; $\lambda\nu=-d$; $\alpha=\dfrac{a\lambda+b}{\lambda-\nu}$; $\beta=\dfrac{a\nu+b}{\nu-\lambda}$.

9. $u(x)=e^x$.

Chapter 3

1. $u(x)=x$.

2. $u(x)=\sec^2 x+\dfrac{\lambda}{1-\lambda}\tan 1$.

3. $u(x)=\sec^2 x\tan x-\dfrac{\lambda}{2(1+\lambda)}\tan^2 1$.

4. $u(x)=\cos x+\frac{x}{2}(\lambda^2\pi^2-4\lambda)$.

5. $u(x)=e^x+\lambda x\left(\dfrac{e^2-1}{2(1-2\lambda)}\right)$.

6. $u(x)=A$, a constant.

7. $u(x)=0$.

8. $u(x)=A\dfrac{\sin x}{2}$, where A is an arbitrary constant.

9. $u(x)=A\dfrac{3x}{1000}$, where A is a constant.

10. $u(x)$ does not have a solution.

11. $u(x)=x+\dfrac{\lambda(10+(6+\lambda)x)}{12-24\lambda-\lambda^2}$.

12. $u(x)=x+\dfrac{\lambda(x(6-\lambda)-4)}{12+\lambda^2}$.

14. $u(x)=x+\dfrac{\lambda\pi}{2}\left\{\dfrac{2\pi-\lambda\pi^2+8\lambda}{2-3\lambda\pi+\lambda^2\pi^2}\right\}+\left\{\dfrac{2\lambda\pi\sin x}{2-3\lambda\pi+\lambda^2\pi^2}\right\}$.

15. $u(x)=\sin x$.

16. $u(x)=\sec^2 x$.

17. $u(x) = \dfrac{1}{1+x^2}$.

18. $u(x) = Cx$, where C is an arbitrary constant.

19. $u(x) = C$, where C is an arbitrary constant.

20. $u(x) = C \sec x$, where C is an arbitrary constant.

21. $u(x) = C \sec x$, where C is an arbitrary constant.

22. $u(x) = \frac{2}{\pi - 2} C \sin^{-1} x$, where C is an arbitrary constant.

23. $u(x) = C \cos x$, where C is an arbitrary constant.

24. $u_1(x) = \frac{2}{\pi} C(\sin x + \cos x)$, and $u_2(x) = \frac{2}{\pi} C(\sin x - \cos x)$, where C is an arbitrary constant.

Chapter 4

1(a). $u(x) = x^2$.

1(b). $u(x) = \tan x$.

1(c). $u(x) = e^x$.

2(a). $u(x) = \dfrac{1 \pm \sqrt{1-2\lambda}}{\lambda}, \lambda \leq \dfrac{1}{2}$. Here, $\lambda = 0$ is a singular point, and $\lambda = \dfrac{1}{2}$ is a bifurcation point and at this point $u(x) = 2$.

2(b). $u(x) = \dfrac{2 \pm 2\sqrt{1-\lambda}}{\lambda}, \lambda \leq 1$. Here, $\lambda = 0$ is a singular point, and $\lambda = 1$ is a bifurcation point and this point $u(x) = 2$.

2(c). $u(x) = \sin x$.

2(d). $u(x) = x, x - 1$.

3(a). $u(x) = 1$.

3(b). $u(x) = 1 + \dfrac{\lambda}{4} + \dfrac{\lambda^2}{8} + \cdots$.

3(c). $u(x) = \sinh x$.

3(d). $u(x) = \sec x$.

7. D'Alembert's wave solution is given by

$$u(x,t) = \frac{1}{2}\left[\frac{1}{1+(x-ct)^2} + \frac{1}{1+(x+ct)^2}\right] + \frac{1}{2c}[\tan(x+ct) - \tan(x-ct)].$$

Chapter 5

1(a). $u(x)=2\sqrt{x}+\dfrac{1}{\sqrt{x}}.$

1(b). $u(x)=\dfrac{2\sqrt{x}}{\pi}\left(1+\dfrac{8}{5}x^2\right).$

1(c). $u(x)=\dfrac{2\sqrt{x}}{\pi}.$

1(d). $u(x)=\dfrac{128}{35\pi}x^{\frac{7}{2}}.$

2(a). $u(x)=\sqrt{x}.$

2(b). $u(x)=\dfrac{1}{2}.$

2(c). $u(x)=\dfrac{1}{\sqrt{\pi}}\left(1-e^{\pi x}erfc(\sqrt{\pi x})\right).$

2(d). $u(x)=x.$

2(e). $u(x)=1.$

Chapter 6

1. $u(x)=\frac{4}{27}x^2+\frac{1}{6}.$
2. $u(x)=\sin x.$
3. $u(x)=\sec^2 x.$
4. $u(x)=xe^x.$
5. $u(x)=\sin x.$
6. $u(x)=\sin x.$
7. $u(x)=x\cos x.$
8. $u(x)=\sin x-\cos x.$
9. $u(x)=1-\sinh x.$
10. $u(x)=\frac{1}{2}(\cos x+\sin x+e^x).$
11. $u(x)=1+\sin x.$
12. $u(x)=\frac{1}{4}(e^x-3e^{-x}-2\cos x).$

Chapter 7

3. $G(x,\eta)=\begin{cases} \dfrac{\sin(\eta-1)\sin x}{\sin(1)} & 0\le x\le \eta \\ \dfrac{\sin\eta\sin(x-1)}{\sin(1)} & \eta\le x\le 1 \end{cases}$

   The solution will be
   $y(x)=\int_0^1 G(x,\eta)f(\eta)\,d\eta$ where $G(x,\eta)$ is given above.

4. Given that $Ty''(x)=-f(x)$, where $f(x)=x$. The boundary conditions are $y(0)=y(\ell)=0$. Using Green's function method, the solution of this boundary value problem can be obtained as $y(x)=\int_0^\ell G(x,\eta)f(\eta)d\eta$ provided $G(x,\eta)$ satisfies the following differential equation with its four boundary conditions:

$$\begin{aligned} TG_{xx} &= -\delta(x-\eta) \\ G(0,\eta) &= 0 \\ G(\ell,\eta) &= 0 \\ G(\eta+,\eta) &= G(\eta-,\eta) \\ T(G_x|_{\eta+}-G_x|_{\eta-}) &= -1. \end{aligned}$$

   The solution of the given ODE satisfying the boundary conditions is simply $y(x)=1/6Tx(\ell^2-x^2)$. The Green function is obtained as

$$G(x,\eta)=\frac{1}{T\ell}\begin{cases} x(\ell-\eta), & 0\le x\le\eta \quad \equiv x\le\eta\le\ell \\ \eta(\ell-x), & \eta\le x\le\ell \quad \equiv 0\le\eta\le x. \end{cases}$$

   Thus, the solution by Green's function method is

$$\begin{aligned} y(x) &= \int_0^\ell G(x,\eta)f(\eta)d\eta \\ &= \frac{1}{T\ell}\left\{\int_0^x \eta(\ell-x)(\eta)d\eta + \int_x^\ell x(\ell-\eta)(\eta)d\eta\right\} \\ &= \tfrac{1}{6T}x(\ell^2-x^2). \end{aligned}$$

   It is obvious these two solutions are identical.

5. (i) $G(x,\eta)=\begin{cases} -e^{\eta-x}\cos\eta\sin x & 0\le x\le\eta \\ -e^{\eta-x}\sin\eta\cos x & \eta\le x\le\frac{\pi}{2} \end{cases}$

Not symmetric.

(ii) $G(x,\eta) = \begin{cases} -e^{-\eta-x}\cos\eta\sin x & 0\le x\le\eta \\ -e^{-\eta-x}\sin\eta\cos x & \eta\le x\le\frac{\pi}{2} \end{cases}$

Green's function now symmetric.

6. $G(x,\eta) = \begin{cases} \cos\eta\sin x & 0\le x\le\eta \\ \sin\eta\cos x & \eta\le x\le\pi. \end{cases}$

7. $G(x,\eta) = \begin{cases} e^{\eta}(1-e^{-x}) & 0\le x\le\eta \\ e^{\eta}(1-e^{-\eta}) & \eta\le x\le\pi. \end{cases}$

8. $G(x,\eta) = \begin{cases} (1-\eta)x & 0\le x\le\eta \\ (1-x)\eta & \eta\le x\le 1. \end{cases}$

9. $G(x,\eta) = \begin{cases} x & x\le\eta \\ \eta & x>\eta. \end{cases}$

10. $G(x,\eta) = \begin{cases} -\dfrac{\cos\lambda(\eta-x-\frac{1}{2})}{2\lambda\sin\frac{\lambda}{2}} & 0\le x\le\eta \\ -\dfrac{\cos\lambda(x-\eta-\frac{1}{2})}{2\lambda\sin\frac{\lambda}{2}} & \eta\le x\le 1. \end{cases}$

11. (a) $G(x,\eta) = \begin{cases} \dfrac{\sin\lambda x\cos\lambda(b-\eta)}{\lambda\cos\lambda b} & 0\le x\le\eta \\ \dfrac{\sin\lambda\eta\cos\lambda(b-x)}{\lambda\cos\lambda b} & \eta\le x\le b \end{cases}$

where $\lambda \neq \dfrac{(2n-1)\pi}{b}$.

(b) $G(x,\eta) = \begin{cases} \dfrac{\cos\lambda x\sin\lambda(b-\eta)}{\lambda\cos\lambda b} & 0\le x\le\eta \\ \dfrac{\cos\lambda\eta\sin\lambda(b-x)}{\lambda\cos\lambda b} & \eta\le x\le b \end{cases}$

where $\lambda \neq \dfrac{(2n-1)\pi}{b}$.

(c) $G(x,\eta) = \begin{cases} -\dfrac{\cos\lambda(a-x)\cos\lambda(b-\eta)}{\lambda\sin\lambda(a-b)} & a\le x\le\eta \\ -\dfrac{\cos\lambda(a-\eta)\cos\lambda(b-x)}{\lambda\sin\lambda(a-b)} & \eta\le x\le b. \end{cases}$

$$(d)\ G(x,\eta)= \begin{cases} -\dfrac{\{\lambda\cos\lambda(a-x)-\sin\lambda(a-x)\}\sin\lambda(b-\eta)}{\lambda[\sin\lambda(b-a)-\lambda\cos\lambda(b-a)]} & a\le x\le\eta \\ -\dfrac{\{\lambda\cos\lambda(a-\eta)-\sin\lambda(a-\eta)\}\sin\lambda(b-x)}{\lambda[\sin\lambda(b-a)-\lambda\cos\lambda(b-a)]} & \eta\le x\le b. \end{cases}$$

13. $$(a)\ G(x,\eta)= \begin{cases} -\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) & 0\le x\le\eta \\ -\dfrac{1}{2}\ln\left(\dfrac{1+\eta}{1-\eta}\right) & \eta\le x\le 1. \end{cases}$$

$$(b)\ G(x,\eta)= \begin{cases} \dfrac{\sin\lambda(\eta-1)\sin\lambda x}{\lambda\sin\lambda} & 0\le x\le\eta \\ \dfrac{\sin\lambda\eta\sin\lambda(x-1)}{\lambda\sin\lambda} & \eta\le x\le 1. \end{cases}$$

15. $$G(x,\eta)= \begin{cases} Ax+B \\ -Ax+B \end{cases}$$

Note: Cannot determine the constants with the given conditions.

16. $$G(x,\xi)= \begin{cases} x^3\dfrac{\xi}{2}+x\dfrac{\xi^3}{2}-9x\dfrac{\xi}{5}+x & 0\le x\le\xi \\ x^3\dfrac{\xi}{2}+x\dfrac{\xi^3}{2}-9x\dfrac{\xi}{5}+\xi & \xi\le x\le 1 \end{cases}$$

17. Green's function is

$$G(x,\xi)= \begin{cases} \sin x\cos\xi & x\le\xi \\ \cos x\sin\xi & x\ge\xi \end{cases}$$

So that the solution becomes

$$\begin{aligned} y(x) &= \int_0^x G(x,\xi)f(\xi)d\xi+\int_x^{\pi/2} G(x,\xi)f(\xi)d\xi \\ &= \int_0^x \cos x\sin\xi\, d\xi+\int_x^{\pi/2}\sin x\cos\xi\, d\xi \\ y(x) &= -1+\sin x+\cos x \end{aligned}$$

Chapter 8

1. $u(x) = \frac{1}{3}\left(x + \frac{2}{\sqrt{3}}\sin(\sqrt{3}x)\right)$.

2. $u(x) = \frac{1}{4} + \frac{3}{4}\cosh 2x$.

3. $f(x) = \begin{cases} \sqrt{6}x, & x \geq 0 \\ -\sqrt{6}x, & x < 0. \end{cases}$

4. Given that $f(x) = \frac{2}{\pi}\int_0^{\pi/2} u(x\sin\theta)d\theta$. It can be easily seen that $f(0) = u(0)$. In addition we see that $f'(x) = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta u'(x\sin\theta)d\theta$. Then we have $f'(0) = \frac{2}{\pi}u'(0)$. Hence rewriting in a systematic way

$$\begin{aligned} f'(x) &= \frac{2}{\pi}\int_0^{\pi/2} u'(x\sin\theta)\sin\theta d\theta \\ u(0) &= f(0) \\ u'(0) &= \frac{\pi}{2}f'(0). \end{aligned}$$

Let us write $x\sin\eta$ for x, and we have on multiplying by x and integrating

$$\begin{aligned} & x\int_0^{\pi/2} f'(x\sin\eta)d\eta \\ &= \frac{2x}{\pi}\int_{\eta=0}^{\pi/2}\left\{\int_{\theta=0}^{\pi/2} u'(x\sin\eta\sin\theta)\sin\theta d\theta\right\}d\eta. \end{aligned}$$

Change the order of integration in the repeated integrals on the right-hand side and take new variable ϕ in place of η, defined by the equation

$$\sin\phi = \sin\theta\sin\eta.$$

Then the above integral equation takes the following form:

$$\begin{aligned} & x\int_0^{\pi/2} f'(x\sin\eta)d\eta \\ &= \frac{2x}{\pi}\int_{\theta=0}^{\pi/2}\left\{\int_{\phi=0}^{\theta}\frac{u'(x\sin\phi)\cos\phi d\phi}{\cos\eta}\right\}d\theta \\ &= \frac{2x}{\pi}\int_{\theta=0}^{\pi/2}\left\{\int_{\phi=0}^{\theta}\frac{u'(x\sin\phi)\sin\theta\cos\phi d\phi}{\sqrt{\sin^2\theta - \sin^2\phi}}\right\}d\theta. \end{aligned}$$

Changing the order of integration again, we obtain with the special observation that $0 \leq \theta \leq \pi/2$; $0 \leq \phi \leq \theta \leq \pi/2$,

$$\begin{aligned} x \int_0^{\pi/2} f'(x \sin \eta) d\eta \\ = \frac{2x}{\pi} \int_{\phi=0}^{\pi/2} \left\{ \int_{\theta=\phi}^{\pi/2} \frac{u'(x \sin \phi) \sin \theta \cos \phi d\theta}{\sqrt{\sin^2 \theta - \sin^2 \phi}} \right\} d\phi. \end{aligned}$$

Now it is easy to see that

$$\int_{\theta=\phi}^{\pi/2} \frac{\sin \theta d\theta}{\sqrt{\cos^2 \phi - \cos^2 \theta}} = -\sin^{-1}\left(\frac{\cos \theta}{\cos \phi}\right)\Big|_{\phi}^{\pi/2} = \frac{\pi}{2}.$$

Hence our solution reduces to

$$\begin{aligned} x \int_0^{\pi/2} f'(x \sin \eta) d\eta &= x \int_0^{\pi/2} u'(x \sin \phi) \cos \phi d\phi \\ &= x \int_0^{\pi/2} u'(x \sin \phi) d(\sin \phi) \\ &= u(x \sin \phi)|_0^{\pi/2} \\ &= u(x) - u(0) = u(x) - f(0). \end{aligned}$$

Thus, we must have

$$u(x) = f(0) + x \int_0^{\pi/2} f'(x \sin \theta) d\theta;$$

and it can be verified by substituting that this function is actually a solution.

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